Contents

Chapter 1

Linear Programming

1.1 Introduction

A *linear programming (LP)* problem may be defined as the problem of maximizing or minimizing a *linear function* subject to *linear constraints*. Applications include optimal production plan in manufacturing, optimal allocation of resources, optimal routing, engineering design problems, etc.

The technique of linear programming was developed by Leonid Kantorovich, George B. Dantzig, and John von Neumann.

George B. Dantzig formulated the general linear programming (LP) problem and devised the *simplex method* in 1947. Although several other methods have been developed over the years for solving LP problems, the simplex method continues to be the most efficient and popular method for solving general LP problems, (Rao, 1996).

An example of a linear programming problem is given below:

Example 1.1 *Maximize*

$$f(x_1, x_2) = 2x_1 + x_2 \tag{1.1}$$

subject to

$$
\begin{aligned}
x_1 + x_2 &\leq 6 \\
x_1 + 2x_2 &\leq 8 \\
-x_1 + 3x_2 &\leq 6 \\
x_1 &\geq 0 \\
x_2 &\geq 0
\end{aligned}
\tag{1.2}
$$

1.2 Formulating linear programming problems

Translating real-life problems into mathematical equations of a linear program is the first challenge of this subject. The problem is formulated from a verbal description as follows:

- Identify the decision variables
- Identify the objective function that is to be optimized. For example it may be required to maximize a profit or to minimize a cost.
- Formulate the constraints.
- State other implicit constraints such as non-negativity restrictions.

The example below will illustrate the formulation of a linear programming problem.

Example 1.2 *A small company manufactures two metal products P1 and P2. It will take 4 hours to complete one product P1 and 21 hours for P2. The manufacturing process requires 3 units of metal for P1 and 1 unit for P2. The products are sold and the profit for the products are 20 for P1 and 50 for P2. A stock of 200 metal units is available for the current period and the company wishes to produce a number of products so that it maximizes the profit during 280 hours of work.*

The information can be summarized as in the table below:

Product	*P1*	*P2*	*Limits*
Resources			
Time required	*4*	*21*	*280*
Metal required	*3*	*1*	*200*
Profit	*20*	*50*	

Variables *The decision variables or the unknowns are, in this case, the number of products P1 and P2 that have to be manufactured in given conditions. Let:*
x_1 = *the number of products P1*
x_2 = *the number of products P2*

Objective function *The objective is to maximize profit, which is composed of the sum of the number of products P1 and P2 times the profit per piece:*

$$P(x, y) = 20x_1 + 50x_2 \tag{1.3}$$

Constraints *From the statement of the problem we can identify three types of constraints:*

Time constraints *The time spent on production should be less than 280 hours. Ideally, it is equal to 280, but since the number of products is integer the constraint may not be satisfied.*

Generally, choosing an inequality rather than an equality gives us more flexibility in optimizing the objective. If all the constraints were equality-type, the problem may be over-constrained. When the number of equalities exceeds the number of variables the problem may have no solution.

Because the time required for the production of P1 is 4 hours, the time for P2 is 21 hours, and the maximum time of work is 280 hours, the time constraint is stated as:

$$4x_1 + 21x_2 \leq 280 \tag{1.4}$$

Material constraint *The 200 metal units must be distributed between the two types of product. We shall write the constraint as an inequality for the reasons mentioned before:*

$$3x_1 + x_2 \leq 200 \tag{1.5}$$

Non-negativity constraints *Although it is not clearly stated in the problem, the number of products delivered must be non-negative:*

$$\begin{aligned} x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned} \tag{1.6}$$

Constraints of this type are often called implicit *because they are implicit in the definition of the variables.*

The complete mathematical description of the linear programming problem is:

$$\begin{aligned} \text{Maximize} \quad & P(x_1, x_2) = 20x_1 + 50x_2 \\ \text{subject to} \quad & \\ & 4x_1 + 21x_2 \leq 280 \\ & 3x_1 + x_2 \leq 200 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \end{aligned} \tag{1.7}$$

A *standard LP maximization problem* is stated as:

Find a vector $\mathbf{x} = [x_1, x_2 \ldots, x_n]^T$ to maximize

$$f(x_1, x_2, \ldots, x_n) = c_1x_1 + c_2x_2 + \ldots + c_nx_n \tag{1.8}$$

subject to the constraints:

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n & \leq & b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n & \leq & b_2 \\ & \ldots & \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n & \leq & b_m \end{array} \tag{1.9}$$

$$x_1 \geq 0, \quad x_2 \geq 0, \quad \ldots, \quad x_n \geq 0 \tag{1.10}$$

A *standard LP minimization problem* is stated as:

Find a vector $[y_1, y_2 \ldots, y_m]^T$ to minimize

$$f(y_1, y_2, \ldots, y_n) = b_1y_1 + b_2y_2 + \ldots + b_my_m \tag{1.11}$$

subject to the constraints:

$$
\begin{aligned}
a_{11}y_1 + a_{12}y_2 + \ldots + a_{1m}y_m &\geq c_1 \\
a_{21}y_1 + a_{22}y_2 + \ldots + a_{2m}y_m &\geq c_2 \\
&\ldots \\
a_{m1}y_1 + a_{m2}y_2 + \ldots + a_{nm}y_m &\geq c_n
\end{aligned}
\tag{1.12}
$$

$$y_1 \geq 0, \quad y_2 \geq 0, \quad \ldots, \; y_m \geq 0 \tag{1.13}$$

- The main constraints are written as $\leq$ for the standard maximum problem and $\geq$ for the standard minimum problem.

- If some of the constraints are equalities they should be removed to obtain a standard problem. If $p < m$ constraints are equalities, they can be solved for p of the unknowns and the solution replaced into the objective function and the other constraints. This will reduce the number of variables to $n - p$.

- If a variable x_j is not restricted to be non-negative, it may be replaced by the difference of two non-negative variables $x_j = u_j - v_j$. This adds one variable and two non-negativity constraints to the problem, (Ferguson, 2004).

Example 1.3 *Put the following LP problem into the standard form:*

$$\text{maximize } f(x_1, x_2, x_3) = x_1 + 2x_2 + 3x_3 \tag{1.14}$$

subject to

$$4x_1 + 3x_2 + 2x_3 \leq 10 \tag{1.15}$$

$$x_1 - x_3 = 2 \tag{1.16}$$

$$x_1 + x_2 + x_3 \geq 1 \tag{1.17}$$

$$x_1 \geq 0, \quad x_3 \geq 0 \tag{1.18}$$

Because we have a maximization problem, the inequality (1.17) will be re-written.

The multiplication by -1 will give:

$$-x_1 - x_2 - x_3 \leq -1 \tag{1.19}$$

The relation (1.16) is an equality that has to be removed. We shall replace x_3 from (1.16) into the rest of the problem:

$$\begin{aligned} x_3 &= x_1 - 2 && (1.20)\\ f(x_1, x_2) &= x_1 + 2x_2 + 3(x_1 - 2) = 4x_1 + 2x_2 - 6 && (1.21) \end{aligned}$$

Note that the last term obtained in $f(x_1, x_2)$ may be omitted because the maximum of a function f plus a constant C is the same as the maximum of f.

The constraints are now:

$$4x_1 + 3x_2 + 2(x_1 - 2) \leq 10, \ \textit{or} \ \ 6x_1 + 3x_2 \leq 14 \tag{1.22}$$

$$-x_1 - x_2 - (x_1 - 2) \leq -1, \ \textit{or} \ \ -2x_1 - x_2 \leq -3 \tag{1.23}$$

The problem is not in the standard form yet because there is no non-negativity constraint on x_2. Thus, two new variables will be introduced to replace x_2:

$$x_2 = x_4 - x_5, \ \ \textit{where} \ \ x_4 \geq 0, \ \ x_5 \geq 0 \tag{1.24}$$

and the standard maximization problem is:

$$\text{maximize} \ \ f(x_1, x_4, x_5) = 4x_1 + 2x_4 - 2x_5 \tag{1.25}$$

subject to

$$6x_1 + 3x_4 - 3x_5 \leq 14 \tag{1.26}$$

$$-2x_1 - x_4 + x_5 \leq -3 \tag{1.27}$$

$$x_1 \geq 0, \ \ x_4 \geq 0, \ \ x_5 \geq 0 \tag{1.28}$$

A standard problem can be written in a matrix form if we introduce the notations:

$$\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \dots \\ c_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \tag{1.29}$$

and we obtain the standard maximum problem:

$$\text{maximize } \mathbf{c}^T\mathbf{x} \tag{1.30}$$

subject to

$$\mathbf{A}\mathbf{x} \leq \mathbf{b}, \quad \mathbf{x} \geq 0 \tag{1.31}$$

Definition 1.1 *A vector* $\mathbf{x}$ *is said to be* feasible *if it satisfies the constraints.*

Definition 1.2 *The set of feasible vectors is called the* constraint set.

Definition 1.3 *A linear programming problem is said to be* feasible *if the constraint set is not empty; otherwise it is said to be* infeasible.

Definition 1.4 *The* feasible region *is the set of points that make all linear inequalities constraints true simultaneously.*

1.3 The primal and dual problem

A linear programming problem, referred to as a *primal* problem, has a companion problem associated, called the *dual*.

Definition 1.5 *(Ferguson, 2004) The dual of the standard maximum problem*

$$\textit{maximize } \mathbf{c}^T\mathbf{x} \tag{1.32}$$

$$\textit{subject to } \mathbf{A}\mathbf{x} \leq \mathbf{b}, \quad \mathbf{x} \geq 0 \tag{1.33}$$

is defined to be the standard minimum problem

$$\textit{minimize } \mathbf{b}^T\mathbf{y} \tag{1.34}$$

$$\textit{subject to } \mathbf{A}^{\mathbf{T}}\mathbf{y} \geq \mathbf{c}, \quad \mathbf{y} \geq 0 \tag{1.35}$$

Here $\mathbf{y}$ is used instead of $\mathbf{x}$ as variable vector, $\mathbf{A}$ is an $m \times n$ matrix, $\mathbf{x}$ - an $n \times 1$ vector, $\mathbf{y}$ - an $m \times 1$ vector, $\mathbf{c}$ - an $m \times 1$ vector and $\mathbf{b}$ - an $n \times 1$ vector.

Each maximization problem in LP has its dual, which is a minimizing problem; similarly, each minimizing problem has its corresponding dual, a maximization problem.

Example 1.4 *(Kennedy, 2005)*

Primal :	*maximize*	$3x_1 + 2x_2$	*Dual:*	*minimize*	$4y_1 + 6y_2$
	subject to:	$2x_1 + x_2 \leq 4$		*subject to:*	$2y_1 + 2y_2 \geq 3$
		$2x_1 + 3x_2 \leq 6$			$y_1 + 3y_2 \geq 2$
		$x_1, x_2 \geq 0$			$y_1, y_2 \geq 0$

Example 1.5 *We shall determine the dual problem for Example 1.2 and give an interpretation.*

The primal LP problem is stated as:

$$\begin{aligned} &\text{Maximize} \quad P(x_1, x_2) = 20x_1 + 50x_2 \\ &\text{subject to} \\ &4x_1 + 21x_2 \leq 280 \\ &3x_1 + x_2 \leq 200 \\ &x_1 \geq 0 \\ &x_2 \geq 0 \end{aligned} \tag{1.36}$$

According to the definition, the dual problem is:

$$\begin{aligned} &\text{Minimize} \quad R(y_1, y_2) = 280y_1 + 200y_2 \\ &\text{subject to} \\ &4y_1 + 3y_2 \geq 20 \\ &21y_1 + y_2 \geq 50 \\ &y_1 \geq 0 \\ &y_2 \geq 0 \end{aligned} \tag{1.37}$$

Both the primal and the dual problem can be represented in the table below:

	Product	P1 (x_1)	P2 (x_2)	Limits
Resources				
Time required (y_1)		*4*	*21*	$\leq$ *280*
Metal required (y_2)		*3*	*1*	$\leq$ *200*
Profit		$\geq$ *20*	$\geq$ *50*	

The dual variables may be interpreted as the cost of the resources that are to be involved in the manufacturing process: the cost associated with a unit of time (an hour of work), y_1, and the cost of the material resources (one metal unit), y_2.

The objective would be to minimize the total cost of production during 280 hours of work and using 200 units of metal, and is described by $R(y_1, y_2)$.

The constraints are expressed now in terms of economic values. For example the first constraint may be translated into: the cost of a piece of P1 should be not less than the cost of 4 hours of work plus the cost of 3 metal units. The non-negativity constraints are natural since prices cannot be negative.

An optimal solution to the dual problem provides a shadow price of the resources allocated.

1.4 Geometrical interpretation

A geometrical interpretation may lead also to a method of solution of a LP problem. The discussion below will concern only problems where the number of unknowns is two for a simple visual representation.

If $ax_1 + bx_2 \leq c$ is a constraint, it can be graphically represented by a half-plane bounded by the line $ax_1 + bx_2 = c$. The intersection of all regions bounded by the constraints will give the feasible region of the LP problem. The feasible region is always a convex set, for a LP problem.

The feasible region in any linear program is called a *polytope* if it is bounded. In a 2D space it is a polygon, in a 3D space, a polyhedron.

According to the *fundamental theorem of linear programming*, if the feasible region to any LP problem has at least one point and is convex and if the objective function has a maximum (or minimum) value within the feasible region, then the maximum (or minimum) will always occur at a corner point in that region.

This statement is represented graphically in Figure 1.1.

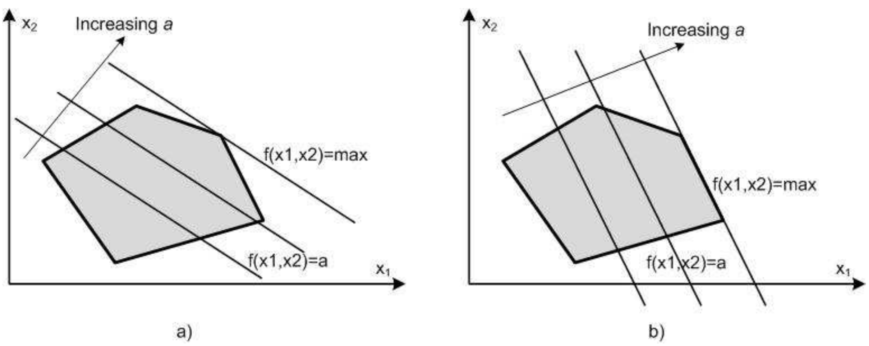

Figure 1.1: Feasible region and the contour lines of f a) One solution, b) Multiple solutions

The contours of the objective function $f(x_1, x_2)$ are straight lines in the $x_1 - x_2$ plane. They are obtained as level curves for $f(x_1, x_2) = a$, where the constant a can take any real value. As a increases, the line will move and the last point where the function line intersects the feasible region is the solution of the problem. Thus, if there is a solution of the problem, it will occur at a vertex.

If the contour lines of f are parallel with one of the constraints line, it is possible to obtain an infinite number of solutions (Figure 1.1 b)) consisting of all the points located on the last edge of intersection with the feasible region, including the two endpoints of the segment.

Example 1.6 *Determine graphically the solution of the following LP problem:*

$$
\begin{aligned}
\text{Maximize} \quad & f(x_1, x_2) = 3x_1 + 4x_2 \\
\text{subject to} \quad & \\
& x_1 + x_2 \leq 6 \\
& x_1 + 2x_2 \leq 8 \\
& -x_1 + 3x_2 \leq 6 \\
& x_1 \geq 0 \\
& x_2 \geq 0
\end{aligned}
\tag{1.38}
$$

The feasible region (Figure 1.2) has been obtained as the intersection of the half-planes bounded by the lines $x_1 + x_2 = 6$, $x_1 + 2x_2 = 8$, $-x_1 + 3x_2 = 6$ and for *non-negative values of the variables.*

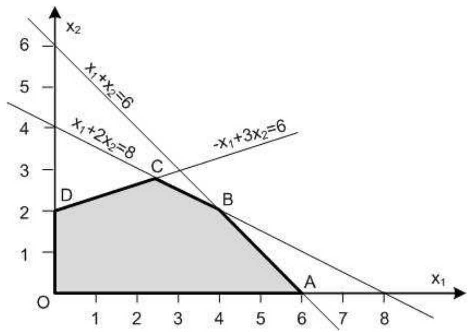

Figure 1.2: Feasible region

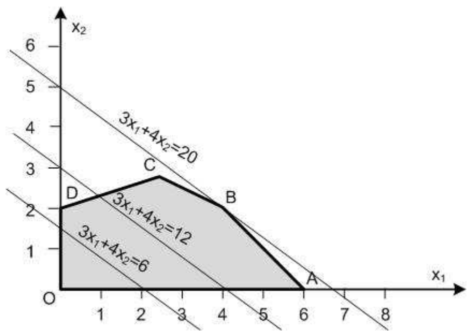

Figure 1.3: Various contour lines of f

The solution of the problem will occur at a vertex of the feasible set, therefore in this case it is enough if we determine all the extreme points of the shaded polygon in Figure 1.2 and choose the one for which f has its maximum value. This approach is not viable if the size of the problem is large (many constraints and many variables).

The points O, A and D can be read directly from the plot, but B and C will be calculated from the intersection of the constraint lines. The point B is the solution of the linear system:

$$\begin{cases} x_1 + x_2 = 6 \\ x_1 + 2x_2 = 8 \end{cases}, \quad x_1 = 4, \;\; x_2 = 2 \tag{1.39}$$

and the point C is the solution of

$$\begin{cases} -x_1 + 3x_2 = 6 \\ x_1 + 2x_2 = 8 \end{cases}, \quad x_1 = \frac{12}{5}, \;\; x_2 = \frac{14}{5} \tag{1.40}$$

All vertices of the feasible region are now determined: O$(0,0)$, A$(6,0)$, B$(4,2)$, C$(12/5,\ 14/5)$ and D$(0,2)$ and the function takes the values:

$$f(0,0) = 0, \;\; f(6,0) = 18, \;\; f(4,2) = 20, \;\; f(\frac{12}{5}, \frac{14}{5}) = \frac{92}{5}, \;\; f(0,2) = 8 \tag{1.41}$$

The maximum occurs at vertex B(4,2) and this is the solution of the problem.

A plot of various contour lines of the objective function is shown in Figure 1.3. The line has such a slope that the last point of intersection with the feasible region is B, the solution of this problem.

1.5 The Simplex algorithm for standard maximization problem

The simplex algorithm of George Dantzig is a popular technique for numerical solution of the LP problems. Although similar in name, it is not related to the downhill simplex method or the Nelder-Mead method.

The method will be described as it is applied to linear programming maximization problems in standard form.

The first step will be to convert inequalities from the constraints into equalities by adding slack variables.

Example 1.7 *If* $2x_1 + x_2 \leq 4$, *a non-negative slack variable* x_3 *will be added and we obtain:* $2x_1 + x_2 + x_3 = 0$, *where* $x_3 \geq 0$.

In general, a vector of non-negative slack variables $\mathbf{x}_{n+j}$, $j = \overline{1, m}$ will be added to the constraints so that the inequalities are written as equalities and the constraints are written in the form:

$$\mathbf{Ax} = \mathbf{b}, \quad \mathbf{x} \geq 0 \tag{1.42}$$

where the augmented decision variable vector is:

$$\mathbf{x} = [x_1 \ \ x_2 \ \ \ldots x_n \ \ x_{n+1} \ \ \ldots x_{n+m}]^T \tag{1.43}$$

The size of vector $\mathbf{b}$ is $m \times 1$, thus m is the number of constraints. In vector x, the variables $x_j, \ j = \overline{1, m}$ are the newly introduced slack variables.

A detailed expression of system (1.42) is:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n + x_{n+1} &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n + x_{n+2} &= b_2 \\ &\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n + x_{n+m} &= b_m \end{aligned} \tag{1.44}$$

and the matrix $\mathbf{A}$ is:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & 1 & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & a_{2n} & 0 & 1 & \dots & 0 \\ \dots & & & & & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} & 0 & 0 & \dots & 1 \end{bmatrix} \tag{1.45}$$

Definition 1.6 *(Ferguson, 2004)*

A basic solution $\mathbf{x}$ *of the system of equations* $\mathbf{A}\mathbf{x} = \mathbf{b}$ *is the solution for which at least* $n - m$ *entries of* $\mathbf{x}$ *are zero.*

Definition 1.7 *(Ferguson, 2004)*

A basic feasible solution (bfs), $\mathbf{x}$ *of the linear programming problem in standard form is a basic solution of the equations* $\mathbf{A}\mathbf{x} = \mathbf{b}$ *for which* $\mathbf{x} \geq 0$

A bfs may be obtained by setting $n - m$ components of $\mathbf{x}$ equal to zero and solving for the remaining m variables. The $n - m$ variables set equal to zero are the *non-basic variables* of the basic solution, the remaining variables are the *basic variables*.

The algorithm will be illustrated by considering a simple example:

Example 1.8

$$\textit{maximize} \quad f(x_1, x_2) = 3x_1 + 2x_2 \tag{1.46}$$

$$\textit{subject to:} \quad 2x_1 + x_2 \leq 4 \tag{1.47}$$

$$x_1 + 2x_2 \leq 4 \tag{1.48}$$

$$x_1, x_2 \geq 0 \tag{1.49}$$

The solution using the simplex method is obtained by the following steps:

Step 1. *Introduce slack variables to convert inequality-type constraints into equalities.*

The non-negative variables x_3 *and* x_4 *will be added to the constraints, and the standard matrix form of the problem is now:*

$$\textit{maximize } f(x_1, x_2) = 3x_1 + 2x_2 \tag{1.50}$$

$$2x_1 + x_2 + x_3 = 4 \quad (1.51)$$

$$x_1 + 2x_2 + x_4 = 4 \quad (1.52)$$

$$x_1, x_2, x_3, x_4 \geq 0 \quad (1.53)$$

In the feasible region of the LP problem, the slack variables will be either positive or zero.

The constraints (1.51), (1.51)are re-written as:

$$x_3 = 4 - 2x_1 - x_2 \quad (1.54)$$

$$x_4 = 4 - x_1 - 2x_2 \quad (1.55)$$

Step 2. *Choose an initial basic feasible solution.*

For this example we may choose the original variables to be zero and the slack x_3 *and* x_4 *are determined from the constraints (1.54), (1.55):*

$$x_1 = 0, \quad x_2 = 0, \quad x_3 = 4, \quad x_4 = 4 \quad (1.56)$$

The value of the objective function is $f = 3x_1 + 2x_2 = 0$.

This bfs is not optimal because a small increase in either the value of x_1 *or* x_2*, so that the non-negativity condition for (1.54), (1.55) still holds will increase the value of the objective function. The simplex algorithm is an iterative method that searches through the basic feasible solutions (bfs) and moves, at each iteration, to a better one in the sense that it has a larger objective function value (for a maximization problem).*

Step 3. *Write the initial tableau.*

In general, if the objective function is written as $f = c_1x_1 + c_2x_2 + \ldots + c_{n+m}x_{n+m}$*, the tableau is:*

	x_1	x_2	...	x_{n+m}	
	a_{11}	a_{12}	...	$a_{1,n+m}$	b_1
	...	...	...	...	...
	a_{m1}	a_{m2}	...	$a_{m,n+m}$	b_m
objective	c_1	c_2	...	c_{n+m}	$-f$

For this example the tableau is:

	x_1	x_2	x_3*	x_4*	
x_3	*2*	*1*	*1*	*0*	*4*
x_4	*1*	*2*	*0*	*1*	*4*
objective	*3*	*2*	*0*	*0*	*0*

The $*$ *and the entries* x_3 *and* x_4 *on the left indicate the basic variables.*

Step 4. *Select the pivot column. This step will identify the non-basic variable to enter the basis.*

The objective function is: $f = 3x_1 + 2x_2 = 0$ *for the current bfs. Any increase of* x_1 *or* x_2 *such that the variables in the basis are non-negative will increase the value of* f*. Because the coefficient of* x_1 *is greater that the one of* x_2*, it will bring a larger increase of the objective. Thus, we shall choose* x_1 *to enter the basis.*

In the simplex tableau, choose the largest positive number from the last row (objective). If there are more with the same value, choose either one.

If all the numbers in the last row are negative or zero the basic solution is the optimal one and the algorithm will stop here.

	x_1	x_2	x_3*	x_4*	
x_3	*2*	*1*	*1*	*0*	*4*
x_4	*1*	*2*	*0*	*1*	*6*
objective	*3*	*2*	*0*	*0*	*0*
	$\uparrow$				

Step 5. *Select the pivot row. This step will identify the basic variable to leave the basis. The intersection of pivot row and pivot column is the pivot element or simply the* pivot*. It must always be a positive number.*

If we keep $x_2 = 0$ *and increase* x_1*, the basic variables are:* $x_3 = 4 - 2x_1$*,* $x_4 = 4 - x_1$*. The variable* x_3 *becomes negative as* x_1 *passes through* 2 *and* x_4 *becomes negative when* x_1 *increases more than* 4*. Thus, the largest value* x_1 *can take so the solution is still feasible is* $x_1 = 2$*.*

In the simplex tableau the reasoning above is translated as: in the pivot column j*, the pivot will be the element which minimizes the ratio* b_k/a_{kj} *over those rows for which* $a_{ij} > 0$*.*

If all elements in the pivot column are negative or zero ($a_{kj} \leq 0,\ k = \overline{1,m}$) then the problem is unbounded above (the maximum of the problem is infinity).

	x_1	x_2	x_3*	x_4*		
x_3	$\boxed{2}$	1	1	0	4	$(b_1/a_{11} = 4/2 = 2 \rightarrow$ *minimum*)
x_4	1	2	0	1	4	$(b_2/a_{21} = 4/1 = 4)$
objective	3	2	0	0	0	
	$\uparrow$					

The pivot is boxed in the tableau above. The variable x_1 enters the basis and the variable x_3 leaves the basis.

Step 6. *Perform the pivot operation, when the pivot element is a_{ij}. This is the process of rewriting the problem in terms of the new basic variables.*

The description of this operation in equations is as follows:

- *The first basic variables were x_3 and x_4:*

$$x_3 = 4 - 2x_1 - x_2 \tag{1.57}$$

$$x_4 = 4 - x_1 - 2x_2 \tag{1.58}$$

$$f = 3x_1 + 2x_2 \tag{1.59}$$

- *Divide (1.57) by 2 (the coefficient of x_1) and rearrange to get x_1, then substitute x_1 in (1.58) and (1.59):*

$$x_1 = 2 - \frac{1}{2}x_2 - \frac{1}{2}x_3 \tag{1.60}$$

$$x_4 = 4 - (2 - \frac{1}{2}x_2 - \frac{1}{2}x_3) - 2x_2 = 2 - \frac{3}{2}x_2 + \frac{1}{2}x_3 \tag{1.61}$$

$$f = 3(2 - \frac{1}{2}x_2 - \frac{1}{2}x_3) + 2x_2 = 6 + \frac{1}{2}x_2 - \frac{3}{2}x_3 \tag{1.62}$$

The corresponding procedure in the simplex tableau is to:

- *Divide the pivot row i by the pivot a_{ij}*
- *add $-a_{kj}/a_{ij} \times row(i)$ to row k for each $k \neq i$ (including objective row). Each element in the rows (non-pivot) will be added by the element in the same row and pivot column divided by the pivot and multiplied by the element in the same column and pivot row.*

	x_1*	x_2	x_3	x_4*	
x_1	*1*	$\frac{1}{2}$	$\frac{1}{2}$	*0*	*2*
x_4	$1-\frac{1}{2}\cdot 2$	$2-\frac{1}{2}\cdot 1$	$0-\frac{1}{2}\cdot 1$	$1-\frac{1}{2}\cdot 0$	$4-\frac{1}{2}\cdot 4$
objective	$3-\frac{3}{2}\cdot 2$	$2-\frac{3}{2}\cdot 1$	$0-\frac{3}{2}\cdot 1$	$0-\frac{3}{2}\cdot 0$	$0-\frac{3}{2}\cdot 4$

or:

	x_1*	x_2	x_3	x_4*	
x_1	*1*	$\frac{1}{2}$	$\frac{1}{2}$	*0*	*2*
x_4	*0*	$\frac{3}{2}$	$-\frac{1}{2}$	1	*2*
objective	*0*	$\frac{1}{2}$	$-\frac{3}{2}$	*0*	*-6*

Step 7. *Go to Step 4 until the basic feasible solution is optimal. The algorithm will stop when all the elements in the last row (objective) are negative or zero. The bottom right entry, which is $-f$ will not be included in this test.*

For the given example the stop criterion is not fulfilled thus we return at step 4 and identify the pivot column and row. The only positive element on the last row is $1/2$ so the second column is the pivot column.

	x_1*	x_2	x_3	x_4*	
x_1	*1*	$\frac{1}{2}$	$\frac{1}{2}$	*0*	*2 (2/*$\frac{1}{2}=4$*)*
x_4	*0*	$\boxed{\frac{3}{2}}$	$-\frac{1}{2}$	1	*2 (2/*$\frac{3}{2}=4/3\rightarrow$ *minimum)*
objective	*0*	$\frac{1}{2}$	$-\frac{3}{2}$	*0*	*-6*
		$\uparrow$			

The current bfs is not optimal, x_2 will enter the basis and x_4 will leave. The pivot operation gives:

	x_1*	x_2*	x_3	x_4	
x_1	*1*	*0*	$\frac{2}{3}$	$-\frac{1}{2}$	$\frac{4}{3}$
x_2	*0*	*1*	$-\frac{1}{3}$	$\frac{2}{3}$	$\frac{4}{3}$
objective	*0*	*0*	$-\frac{4}{3}$	$-\frac{1}{3}$	$-\frac{20}{3}$

Since the elements in the last row are all non-positive, the optimal solution is:

$$x_1 = \frac{4}{3}, \;\; x_2 = \frac{4}{3}, \;\; x_3 = x_4 = 0 \tag{1.63}$$

The optimal value of the objective function is $20/3$, obtained as minus the bottom-right entry of the tableau.

1.6 Exercises

1. Formulate mathematically the following LP problem:

   A plant processes two chemicals A and B. It takes 6 days and 3 kilograms of raw material to make one kilogram of product A and 3 days and 2 kilograms of raw material to make one kilogram of B. The company can sell the product A for \$20/kg and the product B for \$15/kg. Which is the optimal quantity of each product the company would process in three months (90 days) in order to maximize the profit?

2. Formulate mathematically the following LP problem, (Page, 2007):

   A plant makes aluminum and copper wire. Each pound of aluminum wire requires 5 kwh of electricity and 1/4 hr of labor. Each pound of copper wire requires 2 kwh of electricity and 1/2 hr of labor. Production of copper wire is restricted by the fact that raw materials are available to produce at most 60 lbs/day. Electricity is limited to 500 kwh/day and labor to 40 personhrs/day. If the profit from aluminum wire is \$0.25/lb and the profit from copper is \$0.40/lb., how much of each should be produced to maximize profit and what is the maximum profit?

3. Consider the problem:

   $$\text{Maximize } 3x_1 + 2x_2 + x_3 \tag{1.64}$$

   subject to

   $$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0 \tag{1.65}$$

   and

   $$\begin{aligned} x_1 - x_2 + x_3 &\leq 4 \\ 2x_1 + x_2 + 3x_3 &\leq 6 \\ -x_1 + 2x_3 &\leq 3 \\ x_1 + x_2 + x_3 &\leq 8 \end{aligned} \tag{1.66}$$

   State the dual minimum problem

4. Solve graphically the problem from Example 1.8.

5. Solve the following LP problem by inspecting the vertices of the feasible region:

   Maximize
$$f(x,y) = 143x + 60y \tag{1.67}$$
   subject to the constraints:
$$\begin{aligned} x + y &\leq 100 \\ 120x + 210y &\leq 15000 \\ 110x + 30y &\leq 4000 \\ x, y &\geq 0 \end{aligned} \tag{1.68}$$

6. Minimize
$$f(x,y) = 60x + 30y \tag{1.69}$$
   subject to the constraints:
$$\begin{aligned} 2x + 3y &\geq 120 \\ 2x + y &\geq 80 \\ x, y &\geq 0 \end{aligned} \tag{1.70}$$
   Check the vertices to find that the minimum value is 2400 at (0,80) and (30,20).

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