Chapter 3

Stability, controllability, observability

3.1 Stability analysis

In Chapter 2 we have already seen that the values of the system's poles are equal to the eigenvalues of the system matrix A and they do not depend on the basis used for state variables. This will be illustrated in the following example:

Example 3.1 *Given the system represented in spate-space by:*

$$\begin{aligned} \dot{\boldsymbol{x}} &= \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} \boldsymbol{x} \end{aligned}$$

we will determine the eigenvalues of the system matrix, the transfer function and the poles of the transfer function.

- *The eigenvalues of the matrix A are obtained from:*

$$(\lambda I - A) = \begin{bmatrix} \lambda & -1 \\ 2 & \lambda + 3 \end{bmatrix}$$
$$\det(\lambda \mathbf{I} - \mathbf{A}) = \lambda(\lambda + 3) + 2 = (\lambda + 1)(\lambda + 2) = 0$$

and result as:

$$\lambda_1 = -1, \quad \lambda_2 = -2$$

- *The system transfer function is computed from*

$$\begin{aligned} H(s) &= C(sI - A)^{-1}B \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \frac{1}{(s+1)(s+2)} \end{aligned}$$

The poles are -1 *and* -2*, the same as the eigenvalues of the system matrix* A*.*

Note however that this does not hold for all cases: it is possible that a part of the system poles are canceled by some of the system zeros when transforming a state-space model into a transfer function.

Example 3.2 *Consider a system with the input* u *and the output* y *having the state-space model:*

$$\begin{aligned} \dot{\boldsymbol{x}} = A\boldsymbol{x} + Bu \quad &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y = C\boldsymbol{x} \quad &= \begin{bmatrix} -1 & 1 \end{bmatrix} \boldsymbol{x} \end{aligned}$$

The characteristic equation is:

$$\begin{aligned} \det(sI - A) &= \det\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) \\ &= \det \begin{bmatrix} s & -1 \\ -1 & s \end{bmatrix} \\ &= s^2 - 1 \\ &= (s-1)(s+1) \end{aligned}$$

The eigenvalues of the system matrix, or the system poles are $\lambda_1 = 1$ *and* $\lambda_2 = -1$*.*

The transfer function is computed from:

$$\begin{aligned} H(s) &= C(sI - A)^{-1}B \\ &= \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} s & -1 \\ -1 & s \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{s}{(s-1)(s+1)} & \frac{1}{(s-1)(s+1)} \\ \frac{1}{(s-1)(s+1)} & \frac{s}{(s-1)(s+1)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{aligned}$$

or

$$H(s) = \frac{s-1}{(s-1)(s+1)} = \frac{1}{s+1}$$

The transfer function has only one pole at -1 *because the other pole* ($s = 1$) *was canceled with the zero at* $s = 1$*.*

A state-space formulation will give more information about the system than the input-output formulation described by a transfer function. The (internal) system poles, that are the eigenvalues of the system matrix, should be distinguished from the poles of the transfer function as there may be some pole/zero cancelation in computing the transfer function. Then, the denominator of the transfer function is not the same as $\det(sI - A)$ since some poles do not occur in the transfer function (Lewis, 2008).

The concept of stability will be modified to differentiate *internal stability* (given by the eigenvalues of the system matrix) from *external stability* (given by the transfer function poles).

An outline of the stability conditions is given in Table 3.1 in terms of system poles or eigenvalues of the system matrix: $\lambda_i = \sigma_i + j\omega_i$.

Stability condition	**Root values**
Stable	$\sigma_i < 0$, for all $i = \overline{1,n}$ (All the roots are in the left-half s-plane)
Marginally stable	$\sigma_i = 0$ for any i for simple roots and no $\sigma_i > 0$, for $i = \overline{1,n}$ (At least one simple root and no multiple roots on the $j\omega$ axis and the other roots on the left-half s-plane)
Unstable	$\sigma_i > 0$ for any i or $\sigma_i = 0$ for any multiple-order root, $i = \overline{1,n}$ (At least one simple root in the right-half s-plane or at least one multiple-order root on the $j\omega$ axis)

Table 3.1: Stability conditions for linear continuous systems, (Golnaraghi and Kuo, 2010)

When we use state-space descriptions of dynamic systems we discuss the following types of stability:

- **Internal stability** that refers to the state variables. The stability conditions in Table 3.1 are applied for the eigenvalues of system matrix A.

- **External stability** that refers to the output signal. The stability conditions in Table 3.1 are applied to the poles of the transfer function poles $H(s)$.

Transfer functions can only be used to determine the external stability of systems. State-space descriptions can be used to analyze both internal and external stability. It is possible for systems to have external stability but not internal stability.

Example 3.3 *Consider again the system given in Example 3.2. The system is externally stable because it has one negative pole $p = -1$ of the transfer function. It has, however, two eigenvalues of the matrix A: $\lambda_1 = 1$ and $\lambda_2 = -1$. Since one of them is* *positive ($\lambda_1 > 0$), the system is internally unstable. Indeed, if we compute the states using for example the Laplace transform method we obtain:*

$$\boldsymbol{x}(s) = (sI - A)^{-1}BU(s) = \begin{bmatrix} \frac{s}{(s-1)(s+1)} & \frac{1}{(s-1)(s+1)} \\ \frac{1}{(s-1)(s+1)} & \frac{s}{(s-1)(s+1)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} U(s) = \begin{bmatrix} \frac{1}{(s-1)(s+1)} \\ \frac{s}{(s-1)(s+1)} \end{bmatrix} U(s)$$

For example, for an ideal impulse input $u = \delta$ or $U(s) = 1$ the states are:

$$\boldsymbol{x} = \pounds^{-1}\boldsymbol{x}(s) = \begin{bmatrix} \pounds^{-1}\left\{\frac{1}{2}\left(\frac{1}{s-1} - \frac{1}{s+1}\right)\right\} \\ \pounds^{-1}\left\{\frac{1}{2}\left(\frac{1}{s-1} + \frac{1}{s+1}\right)\right\} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(e^t - e^{-t}) \\ \frac{1}{2}(e^t + e^{-t}) \end{bmatrix}$$

Both states involve an exponential term e^t that increases indefinitely towards infinity for an ideal impulse input, therefore the states are unstable.

The output is obtained as a function of s:

$$Y(s) = \begin{bmatrix} -1 & 1 \end{bmatrix} \boldsymbol{x}(s) = -X_1(s) + X_2(s)$$

or, in time domaine:

$$y = -x_1 + x_2 = -\frac{1}{2}(e^t - e^{-t}) + \frac{1}{2}(e^t + e^{-t}) = \frac{1}{2}(-e^t + e^{-t} + e^t + e^{-t}) = e^{-t}$$

For an impulse input, the output will approach zero in steady-state, thus the system is externally stable.

todo: examples for stability

In the discrete-time case, similar stability conditions can be defined. An outline of the conditions is given in Table 3.1 in terms of system poles or eigenvalues of the system matrix: $\lambda_i = \sigma_i + j\omega_i$.

Stability condition	Root values
Stable	$\lvert\lambda_i\rvert < 1$, for all $i = \overline{1,n}$ (All the roots in the unit circle)
Marginally stable	$\lvert\lambda_i\rvert = 1$ for any i for simple roots and no $\lvert\lambda_i\rvert > 1$, for $i = \overline{1,n}$ (At least one simple root and no multiple roots on the unit circle and the other roots are in the unit circle)
Unstable	$\lvert\lambda_i\rvert > 1$ for any i or $\lvert\lambda_i\rvert = 1$ for any multiple-order root, $i = \overline{1,n}$ (At least one simple root outside the unit circle or at least one multiple-order root on the unit circle)

Table 3.2: Stability conditions for linear discrete-time systems

Again, one can distinguish between internal stability (referring to the eigenvalues of the system matrix) and external stability (poles of the transfer function in the $\mathcal{Z}$ domain).

Remark: If the LTI system is an approximation of a nonlinear system around an equilibrium point, then, based on the stability analysis of the LTI system the following conclusions can be drawn for the nonlinear one:

- LTI system stable $\Rightarrow$ the equilibrium point is locally asymptotically stable
- LTI system unstable $\Rightarrow$ the equilibrium point is unstable

Since a nonlinear system may have several equilibrium points the conclusion only refers to the one analyzed. Furthermore, if the LTI system approximation is marginally stable, the equilibrium point of the nonlinear system may be either stable or unstable, as illustrated in the following examples.

Example 3.4 *Consider the nonlinear system with the dynamics*

$$\begin{aligned} \dot{x} &= x^2 \\ y &= x \end{aligned}$$

The only equilibrium point in $x_0 = 0$. In this point, the LTI approximation of the nonlinear system is

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

where $A = 0$, $B = 0$, $C = 1$. The single eigenvalue of the system matrix is $\lambda_1 = 0$, thus the LTI system is marginally stable. In fact, since the dynamics are given by $\dot{x} = 0$, the state will remain at the initial value, irrespective of this value. However, the equilibrium point of the nonlinear system is unstable: for any $x_0 > 0$, the derivative is positive and the value of the state increases exponentially (see Figure **??***).*

Example 3.5 *On the other hand consider the dynamics*

$$\begin{aligned} \dot{x} &= -x^3 \\ y &= x \end{aligned}$$

Again, the only equilibrium point in $x_0 = 0$, and the LTI approximation is the same as in the previous example, i.e.,

$$\begin{aligned} \dot{x} &= 0 \\ y &= x \end{aligned}$$

and is marginally stable. The equilibrium point of the nonlinear system is stable, (for a trajectory see Figure **??***), provable with Lyapunov's direct method.*

3.2 Controllability, stabilizability, reachability

The time-domain representation of a system, expressed in terms of state variables, can also be used to design a suitable compensation scheme for a control system. We are interested in controlling the system with a control signal $\mathbf{u}$ that is a function of several measurable state variables. An important question is whether this is possible, and the property is called *controllability* or *reachability*. If a system is controllable, and all the state variables are measured we can utilize them in a *full-state feedback control law* as shown in Figure 3.1, (Dorf and Bishop, 2008).

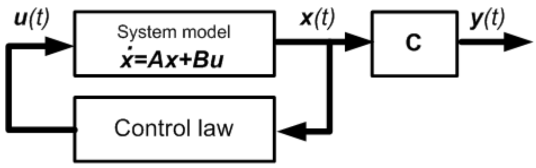

Figure 3.1: Full state feedback control system

In what follows, we first give the definitions and then a simple algebraic test for controllability of an LTI system.

Definition 3.1 (Controllability) *(Graham C. Goodwin, 2000) A state $\boldsymbol{x}_0$ is controllable if there exists a finite interval $[0, T]$ and an input $\{\boldsymbol{u}(t), t \in [0, T]\}$ so that $\boldsymbol{x}(T) = 0$. If all states are controllable, then the system is said to be completely controllable.*

Definition 3.2 (Reachability) *(Graham C. Goodwin, 2000) A state $\bar{\boldsymbol{x}} \neq 0$ is reachable (from the origin) if, given $\boldsymbol{x}(0) = 0$, there exists a finite interval $[0, T]$ and an input $\{\boldsymbol{u}(t), t \in [0, T]\}$ so that $\boldsymbol{x}(T) = \bar{\boldsymbol{x}}$. If all states are reachable, then the system is said to be completely reachable.*

Remark: For continuous-time systems the two definitions above are equivalent, but in discrete-time system there is a difference between them. For instance the system $\boldsymbol{x}(k+1) = 0$ is completely controllable (per the definition above), but no non-zero state is reachable.

Simply put, a system is *controllable* if it is possible to transfer the system from any initial state to any other state in finite time, by means of an unconstrained control vector $\mathbf{u}$, (Ogata, 2002).

Definition 3.3 (Stabilizability) *A system is stabilizable if the uncontrollable subspace is stable.*

A simple algebraic test for determining the controllability (reachability) of an LTI system is the following:

Theorem 3.1 *The LTI system*

$$\begin{aligned}\dot{\boldsymbol{x}} &= A\boldsymbol{x} + B\boldsymbol{u}\\ \boldsymbol{y} &= C\boldsymbol{x}\end{aligned}$$

in continuous time or

$$\begin{aligned}\boldsymbol{x}(k+1) &= A\boldsymbol{x}(k) + B\boldsymbol{u}(k)\\ \boldsymbol{y}(k) &= C\boldsymbol{x}(k)\end{aligned} \tag{3.1}$$

is completely controllable if the controllability matrix

$$\Gamma_c(A,B) = [B\; AB\; A^2B\; \dots\; A^{(n-1)}B] \tag{3.2}$$

has full row rank. The set of all controllable states is given by the range space of $\Gamma_c(A,B)$.

Proof: A sketch of proof for discrete-time LTI systems is given as follows. Given $\boldsymbol{x}(0)$, by iterating on (3.1) we obtain:

$$\begin{aligned}\boldsymbol{x}(1) &= A\boldsymbol{x}(0) + B\boldsymbol{u}(0)\\ \boldsymbol{x}(2) &= A\boldsymbol{x}(1) + B\boldsymbol{u}(1)\\ &= A^2\boldsymbol{x}(0) + AB\boldsymbol{u}(0) + B\boldsymbol{u}(1)\\ \boldsymbol{x}(3) &= A\boldsymbol{x}(2) + B\boldsymbol{u}(2)\\ &= A^3\boldsymbol{x}(0) + A^2B\boldsymbol{u}(0) + AB\boldsymbol{u}(1) + B\boldsymbol{u}(2)\\ &\vdots\\ \boldsymbol{x}(n) &= A^n\boldsymbol{x}(0) + A^{n-1}B\boldsymbol{u}(0) + \cdots + AB\boldsymbol{u}(n-2) + B\boldsymbol{u}(n-1)\end{aligned}$$

which can be written as

$$\boldsymbol{x}(n) = A^n\boldsymbol{x}(0) + \begin{pmatrix}B & AB & A^2B & \dots & A^{n-1}B\end{pmatrix}\begin{pmatrix}\boldsymbol{u}(n-1)\\ \boldsymbol{u}(n-2)\\ \boldsymbol{u}(n-3)\\ \vdots\\ \boldsymbol{u}(0)\end{pmatrix} \tag{3.3}$$

The system (3.8) has a solution $\begin{pmatrix}\boldsymbol{u}(0)^T & \dots & \boldsymbol{u}(n-2)^T & \boldsymbol{u}(n-1)^T\end{pmatrix}^T$ if and only if the matrix $\begin{pmatrix}B & AB & A^2B & \dots & A^{n-1}B\end{pmatrix}$ has full row rank.

Remark: Note that due to the Cayley-Hamilton theorem (see Appendix A), there is no need to consider higher powers of A.

Example 3.6 *Consider a continuous-time LTI system with matrices*

$$A = \begin{pmatrix}-1 & 1\\ 2 & 0\end{pmatrix} \qquad B = \begin{pmatrix}1\\ -3\end{pmatrix}$$

The controllability matrix is

$$\Gamma_c(A,B) = \begin{pmatrix} 1 & -4 \\ -3 & 2 \end{pmatrix}$$

with rank$(\Gamma_c(A,B)) = 2$*, thus the system is completely controllable.*

Example 3.7 *Consider a continuous-time LTI system with matrices*

$$A = \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The controllability matrix is

$$\Gamma_c(A,B) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

with rank$(\Gamma_c(A,B)) = 1$*, thus the system is* not *completely controllable (only one of the states is controllable).*

As already discussed in Chapter 2, LTI systems can be transformed. In fact, for any LTI system there exists a transformation matrix T so that $\bar{A} = T^{-1}AT$, $\bar{B} = T^{-1}B$, $\bar{\boldsymbol{x}} = T^{-1}\boldsymbol{x}$, with the transformed matrices having the form

$$\bar{A} = \begin{pmatrix} A_c & A_{12} \\ 0 & A_{nc} \end{pmatrix} \qquad \bar{B} = \begin{pmatrix} B_c \\ 0 \end{pmatrix}$$

where (A_c, B_c) is completely controllable and A_c has dimension rank$(\Gamma_c(A,B))$.

Furthermore, if a single-input single output (SISO) LTI system is completely reachable, then there exists a transformation to obtain the *controllable (or controllability) canonical form*

$$\bar{A} = \begin{pmatrix} 0 & 0 & 0 & \dots & -a_0 \\ 1 & 0 & 0 & \dots & -a_1 \\ 0 & 1 & 0 & \dots & -a_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -a_{n-1} \end{pmatrix} \qquad B = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{3.4}$$

where $\det(sI - A) = s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0$ is the characteristic polynomial of A. Moreover, this transformation is given by $T = \Gamma_c(A,B)$.

Similarly, for a completely reachable SISO LTI system there exits a similarity transformation that converts the system matrices to the *controller canonical form*

$$\bar{A} = \begin{pmatrix} -a_{n-1} & -a_{n-2} & -a_{n-3} & \dots & -a_0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{3.5}$$

where $\det(sI - A) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$ is the characteristic polynomial of A. This transformation is given by $T = \Gamma_c(A, B)M$, with the matrix M defined as

$$M = \begin{pmatrix} 1 & a_{n-1} & \cdots & a_2 & a_1 \\ 0 & 1 & \cdots & a_3 & a_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

Example 3.8 ***todo***

3.3 Observability, reconstructability, detectability

In general not all states are measured. In such a case, provided that the system possesses certain properties, one may construct an observer to estimate the states that are not directly measured. Such a property of a dynamic system is called *observability* or *reconstructability*. In what follows, we first give a formal definition of these properties and then an algebraic test for them .

Definition 3.4 (Observability) *(Graham C. Goodwin, 2000) A state $\boldsymbol{x}_o \neq 0$ is "unobservable" if, given $\boldsymbol{x}(0) = \boldsymbol{x}_o$ and $\boldsymbol{u} = 0\ \forall t \geq 0$, then $y(t) = 0\ \forall t \geq 0$. If there is no nonzero initial state that is unobservable, then the system is completely observable.*

Similarly to reachability, a related concept is *reconstructability*, which concerns reconstruction of the value of the state, based on past measurements. In what follows, we will use the term *observability* for both. Simply put, a system is *observable* if it is possible to determine the state from the available measurements over a finite time interval, (Ogata, 2002).

Definition 3.5 (Detectability) *A system is detectable if the unobservable subspace is stable.*

The observability test for an LTI system is similar to the controllability one and is the following:

Theorem 3.2 *The LTI system*

$$\begin{aligned} \dot{\boldsymbol{x}} &= A\boldsymbol{x} + B\boldsymbol{u} \\ \boldsymbol{y} &= C\boldsymbol{x} \end{aligned}$$

in continuous time or

$$\begin{aligned} \boldsymbol{x}(k+1) &= A\boldsymbol{x}(k) + B\boldsymbol{u}(k) \\ \boldsymbol{y}(k) &= C\boldsymbol{x}(k) \end{aligned} \tag{3.6}$$

is completely observable if the observability matrix

$$\Gamma_o(A,C) = \begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{pmatrix} \tag{3.7}$$

has full column rank. The set of all observable states is given by the range space of $\Gamma_o(A,C)$.

Proof: A sketch of proof for discrete-time LTI systems is given as follows. Given $\boldsymbol{x}(0)$, by iterating on (3.1) we obtain:

$$\begin{aligned}
\boldsymbol{x}(1) &= A\boldsymbol{x}(0) + B\boldsymbol{u}(0) \\
\boldsymbol{x}(2) &= A\boldsymbol{x}(1) + B\boldsymbol{u}(1) \\
&= A^2\boldsymbol{x}(0) + AB\boldsymbol{u}(0) + B\boldsymbol{u}(1) \\
\boldsymbol{x}(3) &= A\boldsymbol{x}(2) + B\boldsymbol{u}(2) \\
&= A^3\boldsymbol{x}(0) + A^2B\boldsymbol{u}(0) + AB\boldsymbol{u}(1) + B\boldsymbol{u}(2) \\
&\vdots \\
\boldsymbol{x}(n) &= A^n\boldsymbol{x}(0) + A^{n-1}B\boldsymbol{u}(0) + \cdots + AB\boldsymbol{u}(n-2) + B\boldsymbol{u}(n-1)
\end{aligned}$$

which can be written as

$$\boldsymbol{x}(n) = A^n\boldsymbol{x}(0) + \begin{pmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{pmatrix} \begin{pmatrix} \boldsymbol{u}(n-1) \\ \boldsymbol{u}(n-2) \\ \boldsymbol{u}(n-3) \\ \vdots \\ \boldsymbol{u}(0) \end{pmatrix} \tag{3.8}$$

The system (3.8) has a solution $\begin{pmatrix} \boldsymbol{u}(0)^T & \dots & \boldsymbol{u}(n-2)^T & \boldsymbol{u}(n-1)^T \end{pmatrix}^T$ if and only if the matrix $\begin{pmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{pmatrix}$ has full row rank.

Example 3.9 *Consider a discrete-time LTI system with matrices*

$$A = \begin{pmatrix} -2 & -2 \\ 1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & -1 \end{pmatrix}$$

The observability matrix is

$$\Gamma_o(A,C) = \begin{pmatrix} C \\ CA \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -3 & -2 \end{pmatrix}$$

with rank$(\Gamma_o(A,C)) = 2$*, thus the system is completely observable.*

Example 3.10 *Consider a continuous-time LTI system with matrices*

$$A = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & -1 \end{pmatrix}$$

The observability matrix is

$$\Gamma_o(A, C) = \begin{pmatrix} C \\ CA \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix}$$

with rank$(\Gamma_o(A, C)) = 1$, *thus the system is not completely observable.*

For any LTI system there exists a transformation matrix T so that $\bar{A} = T^{-1}AT$, $\bar{B} = T^{-1}B$, $\bar{\boldsymbol{x}} = T^{-1}\boldsymbol{x}$, with the transformed matrices having the form

$$\bar{A} = \begin{pmatrix} A_o & 0 \\ A_{2,1} & A_{no} \end{pmatrix} \qquad \bar{C} = \begin{pmatrix} C_o \\ 0 \end{pmatrix}$$

where (A_o, C_o) is completely observable and A_o has dimension rank$(\Gamma_o(A, C))$.

Furthermore, if a single-input single output (SISO) LTI system is completely observable, then there exists a transformation to obtain the *observable canonical form*

$$\bar{A} = \begin{pmatrix} -a_{n-1} & 1 & 0 & \dots & 0 \\ -a_{n-2} & 0 & 1 & \dots & 0 \\ -a_{n-3} & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_0 & 0 & 0 & \dots & 0 \end{pmatrix} \qquad B = \begin{pmatrix} b_{n-1} \\ b_{n-2} \\ b_{n-3} \\ \vdots \\ b_0 \end{pmatrix} \tag{3.9}$$

$$\bar{C} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \end{pmatrix}$$

where $\det(sI - A) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$ is the characteristic polynomial of A.

Example 3.11 ***todo***

Remark: Both for controllability and observability, if the LTI approximation of a nonlinear system in an operating point is controllable (observable), then the operating point of the nonlinear system is locally controllable (observable). However, the reverse is not true: a nonlinear system may be controllable (observable), while its LTI approximation loses the property.

Example 3.12 *truck-trailer system*

3.4 Canonical (Kalman) decomposition

todo

3.5 Duality

todo