Local stabilization of discrete-time nonlinear systems

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Abstract

This paper considers local stability analysis and local stabilization of discrete-time nonlinear systems represented by Takagi-Sugeno fuzzy models. Conditions are established using Lyapunov functions and controller gains that depend also on past samples. Together with the stability and design conditions, an estimate of the region of attraction is also determined. The developed conditions are discussed and illustrated on several examples.

Index Terms

TS models, local stability, stabilization, domain of attraction

I. Introduction

In the last two decades, stability analysis and control synthesis of nonlinear systems have received significant interest. In this paper, we focus on the particular class of Takagi-Sugeno (TS) models [1]. In both continuous and discrete time, the asymptotic stability of TS models has been studied based on Lyapunov's direct method. Classic approaches are based on the use of quadratic Lyapunov functions [2], [3], but more recently non-quadratic Lyapunov functions [4]–[7], sum-of-squares tools [8]–[10] or homogenous polynomial functions [11], [12] have also been used. The overall aim is to develop linear matrix inequality (LMI) or, lately, sum-of-squares conditions that can be solved using available convex optimization algorithms. Thus, there are many automatic methods for analyzing stability or designing controllers for TS models that can be used in an almost plug-and-play fashion.

Since TS models are generally used to represent nonlinear systems in a compact set of the state-space, e.g., by applying the sector nonlinearity approach [13], the results for stability analysis and controller design are necessarily local: they hold in the largest Lyapunov level set included in the considered domain – the domain of attraction (DA). Therefore, considerable research effort has been devoted to maximizing the DA, both in the continuous and in the discrete-time case. However, many issues remain open, as outlined next.

A nonlinear system may have several equilibria included in the modeling region, in which case standard methods will fail as they do not take this possibility into account (i). Thus, in this case it is necessary to determine a region of attraction before or while proving stability/designing controllers. For TS models, this case has rarely been considered (ii). Although the stability of the origin of the nonlinear system is not necessarily related to that of the local matrices of the TS model, most conditions require the stability of these local models. Without that, usually the LMI conditions for establishing stability become unfeasible and no conclusion can be drawn (iii).

Motivated by these open issues, our goal is to provide a systematic methodology to automatically find the DA and design local controllers for a class of discrete-time nonlinear systems represented by TS models. We develop sufficient LMI conditions that find a DA simultaneously with proving the stability of the origin, thereby addressing issue (ii). Since the DA of the origin is found automatically, our method is not susceptible to failure due to multiple equilibria (i). For the same reason, we also do not explicitly require the local models to be stable/stabilizable (iii).

Our approach combines existing tools in the discrete TS framework with the determination of a non-quadratic DA, by using an easy procedure that efficiently includes the membership functions. To obtain less conservative conditions, we consider the use of past samples both in the Lyapunov function and in the controller gains. As shown in [14], the use of well-chosen past samples can lead to a significant improvement of the results. In particular, their use in the domain definition leads to a larger DA. In order to be able to compute the DA we show that a Lyapunov-like positive semidefinite function may be used to prove stability that does not rely on past samples.

For continuous-time TS models, results for local stability have been achieved due to the necessity of dealing with the derivative of the membership functions that appear when using a nonquadratic Lyapunov functions [6], [7], [15] and the results have been extended to a larger class of nonlinear models. In general, for continuous-time systems, many results exist in the literature, see [16] or [17] and the references therein. Results for the discrete-time case are fewer, even though, at least for TS models, powerful tools exist in the literature. Therefore, we consider discrete-time TS models.

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Simultaneously obtaining a DA and a stabilizing control law has first been considered for linear discrete-time systems with actuator and/or state saturations [18], [19], where the stability analysis is performed using quadratic Lyapunov functions while trying to find the maximum admissible quadratic DA. Still with quadratic Lyapunov functions, the DA has been maximized considering polyhedrons [20] or using saturation dependencies [21]. Nonlinear, quantized systems [22]–[24] have recently been considered e.g., in [25] – for linear systems with a Lipschitz nonlinearity – or [26], in case of SISO quasi-LPV systems, but with a linear controller. A discrete-time output feedback controller for continuous-time linear systems – still in the saturation context – has been developed in [27]. Although we do not consider actuator saturation, the problem we address is the same: simultaneously finding the DA and proving stability/ designing a control law, but considering MIMO nonlinear systems represented by TS models.

For TS systems, the motivation of determining – and if possible, maximizing – a DA is usually that the trajectories of the system should not escape the modeling region. In the last couple of years, several such results have been published. [28] proposed the construction of the DA by an iterative procedure. An improvement thanks to the use of homogeneous polynomial parameter-dependent matrices has been obtained in [29] and enhanced in [30]. [31] considers input-to-state stability and the maximization of the largest Lyapunov level set. Estimates using piecewise affine representations have been derived in [32], [33]. Closed-form estimates using fuzzy-polynomial models have been obtained in [34]. Sufficient sum-of-squares conditions for inescapable sets in the case of uncertain systems [35], [36] have been formulated in [37]. Asymptotically exact conditions have been obtained in [38]. However, in all these results, when developing the DA the membership functions themselves are not considered – except for a bound on derivatives as in [29], or the region where the membership functions are valid as in [32] – and an initial Lyapunov function must be found. Although this means that the results are applicable for a class of systems, it will restrict the results that can be obtained for a specific system. We propose a procedure where the DA explicitly depends on the membership functions, thus relaxing existing results. Preliminary results – using single-sum Lyapunov functions and constant R matrices – on this topic have been presented in our previous papers [39], [40]. Here we generalize those results to general fuzzy Lyapunov functions and general non-PDC control laws. We also give a rigorous mathematical proof regarding the use of Lyapunov-like functions to determine the region of attraction using predicted samples. This is particularly important when the Lyapunov function or the control law depends on past samples, in which case the stability analysis and controller synthesis were missing.

The paper is organized as follows. Section II presents some notations and preliminaries. Section III deals with the local stability analysis and Section IV with local stabilization. In each part, the stability and design conditions, respectively, and the associated DA are given. Simulation examples illustrate the efficiency of the proposed methodology, and discussions are also provided.

II. Notation and preliminaries

In this paper we develop sufficient conditions for the local stability and stabilization of nonlinear discrete-time systems represented by Takagi-Sugeno (TS) fuzzy models. Thus, we consider systems of the form

$$\begin{aligned} \boldsymbol{x}(k+1) &= \sum_{i=1}^{r} h_i(\boldsymbol{z}(k))(A_i\boldsymbol{x}(k) + B_i\boldsymbol{u}(k)) \\ &= A_z\boldsymbol{x}(k) + B_z\boldsymbol{u}(k), \end{aligned} \tag{1}$$

where $\boldsymbol{x}$ denotes the state vector, r is the number of rules, $\boldsymbol{z}$ is the scheduling vector, h_i, $i = 1, 2, \ldots, r$ are normalized membership functions, and A_i, $i = 1, 2, \ldots, r$ are the local models. To motivate this research, consider the following example.

Example 1: Consider the nonlinear system:

$$\begin{aligned} x_1(k+1) &= x_1^2(k) \\ x_2(k+1) &= 3x_1(k) + 0.25x_2(k), \end{aligned} \tag{2}$$

with $x_1(k) \in [-a, a]$, $a > 0$ being a parameter that gives the modeling region. It can be easily seen that (2) is locally stable for $x_1(k) \in (-1, 1)$.

On the domain $x_1(k) \in [-a, a]$, the equivalent TS model is

$$\boldsymbol{x}(k+1) = h_1(x_1(k))A_1\boldsymbol{x} + h_2(x_1(k))A_2\boldsymbol{x},$$

with $h_1(x_1) = \frac{a-x_1(k)}{2a}$, $h_2(x_1(k)) = 1 - h_1(x_1(k))$, $A_1 = \begin{pmatrix} -a & 0 \\ 3 & 0.25 \end{pmatrix}$, $A_2 = \begin{pmatrix} a & 0 \\ 3 & 0.25 \end{pmatrix}$.

Although there are many methods in the literature that may be used to analyze stability of the equilibrium point of this TS model, the conditions will naturally depend on the parameter a. If $a < 1$, e.g., $a = 0.9$, stability is easily provable using a common quadratic Lyapunov function. If $a > 1$, since both A_1 and A_2 are unstable, the usual LMI conditions are unfeasible, thus no conclusion can be drawn regarding the stability of the equilibrium point. On the other hand, it is quite clear that the system (2) has a locally asymptotically stable equilibrium point in $(0, 0)^T$. The question is how to include – essentially a domain – condition such that the LMIs become feasible, thus proving local stability. For this particular example, the condition

$x_1^2(k+1) \leq 0.9x_1^2(k)$ – simply imposing that the state decreases – is enough. However, how to find such a condition for a given TS model is far from trivial and thus motivates the research presented hereafter. $\square$

In what follows, 0 and I denote the zero and identity matrices of appropriate dimensions, and a $(*)$ denotes a term induced by symmetry. The subscript $z+m$ (as in A_{z+m} stands for the scheduling vector being evaluated at the current sample plus mth instant, i.e., at $\boldsymbol{z}(k+m)$. We will also make use of the following results:

Lemma 1: [41] Consider a vector $\boldsymbol{x} \in \mathbb{R}^{n_x}$ and two matrices $Q = Q^T \in \mathbb{R}^{n_x \times n_x}$ and $\mathcal{R} \in \mathbb{R}^{m \times n_x}$ such that $\mathrm{rank}(\mathcal{R}) < n_x$. The two following expressions are equivalent:

1) $\boldsymbol{x}^T Q \boldsymbol{x} < 0$, $\boldsymbol{x} \in \{\boldsymbol{x} \in \mathbb{R}^{n_x}, \boldsymbol{x} \neq 0, \mathcal{R}\boldsymbol{x} = 0\}$
2) $\exists \mathcal{M} \in \mathbb{R}^{m \times n_x}$ such that $Q + \mathcal{M}\mathcal{R} + \mathcal{R}^T\mathcal{M}^T < 0$

Property 1: Let A and B be matrices of appropriate dimensions and ranks, with $B = B^T > 0$. Then

$$(A-B)^T B^{-1}(A-B) \geq 0 \Longleftrightarrow A^T B^{-1} A \geq A + A^T - B.$$

Property 2: (S-procedure) Consider matrices $F_i = F_i^T \in \mathbb{R}^{n \times n}$, $\boldsymbol{x} \in \mathbb{R}^n$, such that $\boldsymbol{x}^T F_i \boldsymbol{x} \geq 0$, $i = 1, \ldots, p$, and the quadratic inequality condition

$$\boldsymbol{x}^T F_0 \boldsymbol{x} > 0, \tag{3}$$

$\boldsymbol{x} \neq 0$. A sufficient condition for (3) to hold is: there exist $\tau_i \geq 0$, $i = 1, \ldots, p$, such that

$$F_0 - \sum_{i=1}^{p} \tau_i F_i > 0.$$

Analysis and design for TS models often lead to double-sum negativity problems of the form

$$\boldsymbol{x}^T \sum_{i=1}^{r} \sum_{i=1}^{r} h_i(\boldsymbol{z}(k)) h_j(\boldsymbol{z}(k)) \Gamma_{ij} \boldsymbol{x} < 0, \tag{4}$$

where Γ_{ij}, $i, j = 1, 2, \ldots, r$ are matrices of appropriate dimensions. Sufficient LMI conditions to ensure (4) are:

Lemma 2: [3] The double-sum (4) is negative, if

$$\begin{aligned} &\Gamma_{ii} < 0, \\ &\Gamma_{ij} + \Gamma_{ji} < 0, \quad i, j = 1, 2, \ldots, r, \; i < j. \end{aligned} \tag{5}$$

For the easier notation, in the case of nonquadratic stability and stabilization, we use the multiindex notation from [14], repeated here for convenience:

Definition 1: (Multiple sum) We denote a multiple sum of matrices $P_{i_1 i_2 \ldots i_{n_P}}$, $i_1, \ldots, i_{n_P} = 1, 2, \ldots, r$ with n_P sums evaluated at sample k and depending on values at different samples

$$\begin{aligned} \mathbb{P}_{G_0^P} = &\sum_{i_1=1}^{r} h_{i_1}(\boldsymbol{z}(k+d_1)) \sum_{i_2=1}^{r} h_{i_2}(\boldsymbol{z}(k+d_2)) \ldots \\ &\sum_{i_{n_P}=1}^{r} h_{i_{n_P}}(\boldsymbol{z}(k+d_{n_P})) P_{i_1 i_2 \ldots i_{n_P}}, \end{aligned}$$

where G_0^P is the multiset of delays (sample differences) $G_0^P = \{d_1, d_2, \ldots, d_{n_P}\}$.

(Multiset of delays) G_0^P denotes the multiset containing the sample differences in the multiple sum involving the matrices $P_{i_1 i_2 \ldots i_{n_P}}$, $i_1, \ldots, i_{n_P} = 1, 2, \ldots, r$ at sample k. G_α^P denotes the multiset containing the delays in the sum P at sample $k+\alpha$.

(Cardinality) The cardinality of a multiset G, $|G|$, is defined as the number of elements in G.

(Index set) The index set of a multiple sum $\mathbb{P}_G$ is $\mathbb{I}_G = \{i_j | i_j = 1, 2, \ldots, r, j = 1, 2, \ldots, |G|\}$, the set of all indices that appear in the sum. An element $\boldsymbol{i} \in \mathbb{I}_G$ is a multiindex.

(Multiplicity) The multiplicity of an element x in a multiset G, $\mathbf{1}_G(x)$ denotes the number of times this element appears in the multiset G.

(Union) The union of two multisets G_A and G_B is $G_C = G_A \cup G_B$ such that $\forall x \in G_C$, $\mathbf{1}_{G_C}(x) = \max\{\mathbf{1}_{G_A}(x), \mathbf{1}_{G_B}(x)\}$.

(Intersection) The intersection of two multisets G_A and G_B is $G_C = G_A \cap G_B$ such that $\forall x \in G_C$, $\mathbf{1}_{G_C}(x) = \min\{\mathbf{1}_{G_A}(x), \mathbf{1}_{G_B}(x)\}$.

(Sum) The sum of two multisets G_A and G_B is $G_C = G_A \oplus G_B$ such that $\forall x \in G_C$, $\mathbf{1}_{G_C}(x) = \mathbf{1}_{G_A}(x) + \mathbf{1}_{G_B}(x)$.

(Projection of an index) The projection of the index $\boldsymbol{i} \in \mathbb{I}_{G_A}$, to the multiset of delays G_B, $\mathrm{pr}_{G_B}^{\boldsymbol{i}}$ is the part of the index that corresponds to the delays in $G_A \cap G_B$.

Example 2: Consider the multiple sum

$$\mathbb{X}_{G_0^X} = \sum_{i_1=1}^{r} h_{i_1}(\boldsymbol{z}(k-1)) \sum_{i_2=1}^{r} h_{i_2}(\boldsymbol{z}(k)) \cdot \sum_{i_3=1}^{r} h_{i_3}(\boldsymbol{z}(k)) X_{i_1 i_2 i_3}.$$

The multiset of delays at the current sample k is $G_0^X = \{-1,\, 0,\, 0\}$ and at sample $k+\alpha$ is $G_\alpha^X = \{\alpha-1,\, \alpha,\, \alpha\}$. The cardinality of G_0^X is $|G_0^X| = 3$, the multiplicity of the elements are $\mathbf{1}_{G_0^X}(-1) = 1$, $\mathbf{1}_{G_0^X}(0) = 2$. The multiplicity of an element not in G_0^X is zero, e.g, $\mathbf{1}_{G_0^X}(-2) = 0$. The index set of $\mathbb{X}_G$ is $\mathbb{I}_G = \{i_j | i_j = 1, 2, \ldots, r,\, j = 1, 2, 3\}$. An element of this set, e.g., $\boldsymbol{i} = 123$ is a multiindex that corresponds to $i_1 = 1$, $i_2 = 2$, $i_3 = 3$. The projection of the multiindex $\boldsymbol{i} = 123$ on the multiset of delays $G_C = \{-1,\, 0\}$ is $\mathrm{pr}^{\boldsymbol{i}}_{G_C} = 12$. Note that this is not unique, $\mathrm{pr}^{\boldsymbol{i}}_{G_C} = 13$ is also a valid projection.

Consider now $G_A = \{-1,\, 0,\, 0\}$ and $G_B = \{-2,\, -1,\, 0\}$. Their union is $G_A \cup G_B = \{-2,\, -1,\, 0,\, 0\}$, the intersection is $G_A \cap G_B = \{-1,\, 0\}$, and the sum is $G_A \oplus G_B = \{-2,\, -1,\, -1,\, 0,\, 0,\, 0\}$. □

Remark 1: Note that although we use the expressions "delay" and "multiset of delays", the system we study is in fact not a delayed one. We will use membership functions evaluated at past samples ("delayed membership functions") in the Lyapunov function and the controller gains in order to relax the stability conditions.

III. Local stability analysis

Consider the autonomous TS model, repeated here for convenience:

$$\boldsymbol{x}(k+1) = A_z \boldsymbol{x}(k), \tag{6}$$

or, using the notation in Section II

$$\boldsymbol{x}(k+1) = \mathbb{A}_{G_0^A} \boldsymbol{x}(k), \tag{7}$$

with $G_0^A = \{0\}$, defined on the domain $\boldsymbol{x} \in \mathcal{D}$ including the origin. Note that in what follows, without loss of generality, we consider the analysis of the equilibrium point $\boldsymbol{x} = 0$. Otherwise, a change of variables can be used to shift the equilibrium point in the origin.

Our goal is to develop conditions to determine whether $\boldsymbol{x} = 0$ is a locally asymptotically stable equilibrium point and determine a region of attraction. For this, we assume that

Assumption 1: There exists a domain $\mathcal{D}_R$ and a multiple sum $\mathbb{R}_{G_0^R}$, with $R_{\boldsymbol{i}} = R_{\boldsymbol{i}}^T$, $\boldsymbol{i} \in \mathbb{I}_{G_R}$, so that

$$\begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}^T \mathbb{R}_{G_0^R} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix} \geq 0$$

holds $\forall \boldsymbol{x}(k) \in \mathcal{D}_R$.

Assumption 1 above can always be satisfied, e.g., by choosing $\mathbb{R}_{G_0^R} = R = R^T > 0$. For given matrices $R_{\boldsymbol{i}}$, the domain $\mathcal{D}_R$ depends on the system being analyzed, and in the worst case $\mathcal{D}_R$ is reduced to the equilibrium point.

A. Stability conditions

Let us consider the nonquadratic Lyapunov function $V = \boldsymbol{x}^T(k)\mathbb{P}_{G_0^P}\boldsymbol{x}(k)$, where G_0^P contains the multiindex used in the Lyapunov matrix at time k, together with the constraint $\begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}^T \mathbb{R}_{G_0^R} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}$ between consecutive samples. Then, the following result can be formulated:

Theorem 1: The discrete-time nonlinear model (6) is locally asymptotically stable in the domain $\mathcal{D}_S$, if there exist matrices $P_{\boldsymbol{i}_j^P} = P_{\boldsymbol{i}_j^P}^T$, $\boldsymbol{i}_j^P = \mathrm{pr}^{\boldsymbol{i}}_{G_j^P}$, $M_{\boldsymbol{i}_j^H}$, $\boldsymbol{i}_j^M = \mathrm{pr}^{\boldsymbol{i}}_{G_j^M}$, and $N_{\boldsymbol{i}_j^H}$, $\boldsymbol{i}_j^N = \mathrm{pr}^{\boldsymbol{i}}_{G_j^N}$, $\boldsymbol{i} \in \mathbb{I}_{G_V}$, $j = 0, 1$, and $R_{\boldsymbol{i}_0^R} = R_{\boldsymbol{i}_0^R}^T$, $\boldsymbol{i}_0^R = \mathrm{pr}^{\boldsymbol{i}}_{G_0^R}$, where $G_V = G_0^P \cup G_1^P \cup (G_0^M \oplus G_0^A) \cup (G_0^N \oplus G_0^A) \cup G_0^R$ so that

$$\begin{pmatrix} \mathbb{N}_{G_0^N}\mathbb{A}_{G_0^A} + (*) - \mathbb{P}_{G_0^P} & (*) \\ \mathbb{M}_{G_0^M}\mathbb{A}_{G_0^A} - \mathbb{N}_{G_0^N}^T & \mathbb{P}_{G_1^P} - \mathbb{M}_{G_0^M} - (*) \end{pmatrix} + \mathbb{R}_{G_0^R} < 0, \tag{8}$$

where $\mathcal{D}_S$ is the largest Lyapunov level set included in $\mathcal{D}_R \bigcap \mathcal{D}$.

Proof 1: Consider the Lyapunov function $V(\boldsymbol{x}(k)) = \boldsymbol{x}(k)^T\mathbb{P}_{G_0^P}\boldsymbol{x}(k)$, with $P_{\boldsymbol{i}_j^P} = P_{\boldsymbol{i}_j^P}^T > 0$, $\boldsymbol{i}_j^P = \mathrm{pr}^{\boldsymbol{i}}_{G_j^P}$. The difference in the Lyapunov function is

$$\Delta V = \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}^T \begin{pmatrix} -\mathbb{P}_{G_0^P} & 0 \\ 0 & \mathbb{P}_{G_1^P} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}.$$

Let us assume that there exist matrices $R_{\boldsymbol{i}_0^R} = R_{\boldsymbol{i}_0^R}^T$, $\boldsymbol{i}_0^R = \mathrm{pr}_{G_0^R}^{\boldsymbol{i}}$, and a domain $\mathcal{D}_R$ such that

$$\begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}^T \mathbb{R}_{G_0^R} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix} \geq 0$$

holds $\forall \boldsymbol{x}(k) \in \mathcal{D}_R$. Since $\boldsymbol{x} = 0$ is an equilibrium point of the system (7), $\mathcal{D}_R$ always exists and it includes $\boldsymbol{x} = 0$. Then, we have $\Delta V < 0$ if

$$\Delta V < -\begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}^T \mathbb{R}_{G_0^R} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}$$

i.e., according to Proposition 2 and since $R_{\boldsymbol{i}_0^R}$ are decision variables

$$\begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}^T \begin{pmatrix} -\mathbb{P}_{G_0^P} & 0 \\ 0 & \mathbb{P}_{G_1^P} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix} + \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}^T \mathbb{R}_{G_0^R} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix} < 0.$$

Writing the dynamics of (6) as

$$\begin{pmatrix} \mathbb{A}_{G_0^A} & -I \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix} = 0$$

and using Lemma 1, we have $\Delta V < 0$ if there exist $\mathcal{M}$ so that

$$\begin{pmatrix} -\mathbb{P}_{G_0^P} & 0 \\ 0 & \mathbb{P}_{G_1^P} \end{pmatrix} + \mathbb{R}_{G_0^R} + \mathcal{M} \begin{pmatrix} \mathbb{A}_{G_0^A} & -I \end{pmatrix} + (*) < 0.$$

Choosing $\mathcal{M} = \begin{pmatrix} \mathbb{N}_{G_0^N} \\ \mathbb{M}_{G_0^M} \end{pmatrix}$ leads to (8). Wrt. the domain of attraction, recall that (6) has been defined in the domain $\mathcal{D}$, and that

$$\begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}^T \mathbb{R}_{G_0^R} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix} \geq 0$$

holds in the domain $\mathcal{D}_R$. Thus, convergence is established for every trajectory starting in $\mathcal{D}_S$, where $\mathcal{D}_S$ is the largest Lyapunov level set contained in $\mathcal{D}_R \bigcap \mathcal{D}$. $\square$

B. Complexity analysis

Sufficient LMI conditions can easily be derived for the above conditions. However, to efficiently apply relaxations such as Lemma 2 and to reduce the computational complexity, the delays used in the Lyapunov function and in the Finsler matrices should be suitably chosen. We will look at the terms that appear in G_V.

Let us first consider G_0^R. Since $\mathbb{R}_{G_0^R}$ is completely free and is added to

$$\begin{pmatrix} \mathbb{N}_{G_0^N} \mathbb{A}_{G_0^A} + (*) - \mathbb{P}_{G_0^P} & (*) \\ \mathbb{M}_{G_0^M} \mathbb{A}_{G_0^A} - \mathbb{N}_{G_0^N}^T & \mathbb{P}_{G_1^P} - \mathbb{M}_{G_0^M} - \mathbb{M}_{G_0^M}^T \end{pmatrix} \tag{9}$$

to reduce the number of sums, G_0^R should contain all the indices appearing in (9), thus it will depend on the membership functions. Moreover, since both $\mathbb{M}_{G_0^M}$ and $\mathbb{N}_{G_0^N}$ are multiplied by $\mathbb{A}_{G_0^A}$, the indices can be chosen the same, i.e., $G_0^M = G_0^N$. Thus, G_V becomes $G_V = G_0^P \cup G_1^P \cup (G_0^M \oplus G_0^A)$.

Further, since for classic TS models $G_0^A = \{0\}$, to apply relaxations, $\mathbb{M}_{G_0^M} \mathbb{A}_{G_0^A}$ should contain a double sum in the same sample, for instance $\{0, 0\}$. Choosing $G_0^P = \emptyset$ we recover the quadratic Lyapunov function $V = \boldsymbol{x}^T(k) P \boldsymbol{x}(k)$. Otherwise, since both G_0^P and G_1^P appear in (9), in order to combine at least one of them with $\mathbb{M}_{G_0^M}$ or $\mathbb{M}_{G_0^M} \mathbb{A}_{G_0^A}$ one can choose either $G_0^P = \{0\}$ [42] or $G_0^P = \{-1\}$ [14]. Then we have either

$$\begin{pmatrix} \mathbb{N}_0 \mathbb{A}_0 + (*) - \mathbb{P}_0 & (*) \\ \mathbb{M}_0 \mathbb{A}_0 - \mathbb{N}_0^T & \mathbb{P}_1 - \mathbb{M}_0 - \mathbb{M}_0^T \end{pmatrix} + \mathbb{R}_{G_0^R} < 0, \tag{10}$$

or

$$\begin{pmatrix} \mathbb{N}_0 \mathbb{A}_0 + (*) - \mathbb{P}_{-1} & (*) \\ \mathbb{M}_0 \mathbb{A}_0 - \mathbb{N}_0^T & \mathbb{P}_0 - \mathbb{M}_0 - \mathbb{M}_0^T \end{pmatrix} + \mathbb{R}_{G_0^R} < 0, \tag{11}$$

respectively. Both cases lead to three sums, a double one in 0 and a single one in either 1 (future sample) or -1 (past sample), thus a relaxation scheme can be applied on the double sum. Since G_0^R, as described above contains the same indices, the total number of sums will be three, with a double sum among them. Furthermore, one may add another sum to M and N that will not modify the total number of sums: adding 1 in the case of (10) and -1 for (11), leads to

$$\begin{pmatrix} \mathbb{N}_{0,1}\mathbb{A}_0 + (*) - \mathbb{P}_0 & (*) \\ \mathbb{M}_{0,1}\mathbb{A}_0 - \mathbb{N}_{0,1}^T & \mathbb{P}_1 - \mathbb{M}_{0,1} - \mathbb{M}_{0,1}^T \end{pmatrix} + \mathbb{R}_{G_0^R} < 0 \tag{12}$$

and

$$\begin{pmatrix} \mathbb{N}_{-1,0}\mathbb{A}_0 + (*) - \mathbb{P}_{-1} & (*) \\ \mathbb{M}_{-1,0}\mathbb{A}_0 - \mathbb{N}_{-1,0}^T & \mathbb{P}_0 - \mathbb{M}_{-1,0} - \mathbb{M}_{-1,0}^T \end{pmatrix} + \mathbb{R}_{G_0^R} < 0 \tag{13}$$

respectively.

Now, any known relaxation [43], [44] can be applied. More generally, in order to apply these relaxations, the choice of the index sets has to favor in G_V multiple sums at the same samples.

Regarding the computational complexity of solving the LMI conditions, according to [45], page 18, a realistic approximation of the numerical complexity using the interior-point method is $\mathcal{O}(N_d^{2.1}N_l^{1.2})$, where N_d is the number of scalar decision variables and N_l is the row size of the LMI problem. Under the assumption that $G_0^N = G_0^M$ and that all the indices are included in G_0^R, the number of decision variables is $N_d = 2r^{|G_0^M|} \times n^2 + r^{|G_0^P|} \times \frac{n(n+1)}{2} + r^{|G_0^R|} \times n(2n+1)$ and the number of rows in the LMI problem – constructed as the block-diagonal LMI having all the individual LMIs on the diagonal – is $N_l = 2r^{|G_0^R|}n$.

C. Handling past samples in the Lyapunov function

In what follows we analyze how the past samples in the Lyapunov function can be handled.

Recall that the domain $\mathcal{D}_S$ will be given by the largest Lyapunov level set in $\mathcal{D}$ and $\mathcal{D}_R$, i.e., the domain where Assumption 1 holds. In fact, see also [34], any trajectory that will arrive in $\mathcal{D}_S$ will eventually converge. Consequently, for a given $\boldsymbol{x}(0)$, if there exists $\beta > 1$ so that $\boldsymbol{x}(\beta) \in \mathcal{D}_S$, then $\boldsymbol{x}(k) \to 0$ as $k \to \infty$.

With this in mind, when using a Lyapunov function involving past samples, instead of determining the level set of e.g., $V(\boldsymbol{x}(k)) = \boldsymbol{x}^T(k)\mathbb{P}_{-1}\boldsymbol{x}(k)$ – which would involve setting $\boldsymbol{x}(-1)$ – that is included in $\mathcal{D}_R$, it is possible to consider the level set of $\boldsymbol{x}^T(k+1)\mathbb{P}_0\boldsymbol{x}(k+1)$ that is included in $\mathcal{D}_{R_1}$. This is equivalent to using the Lyapunov function $V(\boldsymbol{x}(k)) = \boldsymbol{x}^T(k)\mathbb{P}_{G_0^P}\boldsymbol{x}(k)$ only for samples $k > \beta$, with $\beta \geq 0$ a finite number.

Alternatively, consider the Lyapunov-like function

$$V(\boldsymbol{x}) = \boldsymbol{x}(k+\beta)^T\mathbb{P}_{G_0^P}\boldsymbol{x}(k+\beta),$$

together with the domain $\mathcal{D}_R$ given by

$$\begin{pmatrix} \boldsymbol{x}(k+\beta) \\ \boldsymbol{x}(k+\beta+1) \end{pmatrix}^T \mathbb{R}_{G_0^R} \begin{pmatrix} \boldsymbol{x}(k+\beta) \\ \boldsymbol{x}(k+\beta+1) \end{pmatrix} > 0$$

$\forall \boldsymbol{x}(k+\beta) \in \mathcal{D}_R$, where G_0^P and G_0^R only contain 0 or positive delays. Note that this is not a Lyapunov function, as, depending on the system considered, $V(\boldsymbol{x}(k)) > 0$, $\forall \boldsymbol{x}(k) \neq 0$ cannot be ensured. However, $V(\boldsymbol{x}(k))$ is positive semidefinite.

The difference in the Lyapunov function is

$$\Delta V = \begin{pmatrix} \boldsymbol{x}(k+\beta) \\ \boldsymbol{x}(k+\beta+1) \end{pmatrix}^T \begin{pmatrix} -\mathbb{P}_{G_0^P} & 0 \\ 0 & \mathbb{P}_{G_1^P} \end{pmatrix} (*).$$

We have $\Delta V < 0$ if – considering the domain $\mathcal{D}_R$ –

$$\Delta V < -\begin{pmatrix} \boldsymbol{x}(k+\beta) \\ \boldsymbol{x}(k+\beta+1) \end{pmatrix}^T \mathbb{R}_{G_0^R} \begin{pmatrix} \boldsymbol{x}(k+\beta) \\ \boldsymbol{x}(k+\beta+1) \end{pmatrix},$$

i.e.,

$$\begin{aligned} &\begin{pmatrix} \boldsymbol{x}(k+\beta) \\ \boldsymbol{x}(k+\beta+1) \end{pmatrix}^T \begin{pmatrix} -\mathbb{P}_{G_0^P} & 0 \\ 0 & \mathbb{P}_{G_1^P} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k+\beta) \\ \boldsymbol{x}(k+\beta+1) \end{pmatrix} \\ &+ \begin{pmatrix} \boldsymbol{x}(k+\beta) \\ \boldsymbol{x}(k+\beta+1) \end{pmatrix}^T \mathbb{R}_{G_0^R} \begin{pmatrix} \boldsymbol{x}(k+\beta) \\ \boldsymbol{x}(k+\beta+1) \end{pmatrix} < 0. \end{aligned}$$

Writing the dynamics of (6) as

$$\begin{pmatrix} \mathbb{A}_{G_0^A} & -I \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k+\beta) \\ \boldsymbol{x}(k+\beta+1) \end{pmatrix} = 0$$

and using Lemma 1, we have $\Delta V < 0$ if there exist $\mathcal{M}$ so that

$$\begin{pmatrix} -\mathbb{P}_{G_0^P} & 0 \\ 0 & \mathbb{P}_{G_1^P} \end{pmatrix} + \mathbb{R}_{G_0^R} + \mathcal{M}\begin{pmatrix} \mathbb{A}_{G_0^A} & -I \end{pmatrix} + (*) < 0.$$

Choosing $\mathcal{M} = \begin{pmatrix} \mathbb{N}_{G_0^N} \\ \mathbb{M}_{G_0^M} \end{pmatrix}$ leads to

$$\begin{pmatrix} \mathbb{N}_{G_0^N} + (*) - \mathbb{P}_{G_0^P} & (*) \\ \mathbb{M}_{G_0^M}\mathbb{A}_{G_0^A} - \mathbb{N}_{G_0^N}^T & \mathbb{P}_{G_1^P} - \mathbb{M}_{G_0^M} - \mathbb{M}_{G_0^M}^T \end{pmatrix} + \mathbb{R}_{G_0^R} < 0. \tag{14}$$

Note that from the point of view of the feasibility of the LMIs, under the same relaxation scheme, condition (14) is equivalent to (8). However, both V and ΔV above are expressed in terms of $\boldsymbol{x}(k+\beta)$, not $\boldsymbol{x}(k)$. Nevertheless,

$$V(\boldsymbol{x}) = \boldsymbol{x}(k+\beta)^T \mathbb{P}_{G_0^P} \boldsymbol{x}(k+\beta)$$

is positive semidefinite and ΔV is negative semidefinite in $\boldsymbol{x}(k)$. Using Theorem 1 from [46], if condition (14) holds, then the trajectories will converge to the largest invariant set in $\Delta V = 0$. Since the largest invariant set is 0, local stability is established.

Regarding the domain of attraction, recall that (6) has been defined in the domain $\mathcal{D}$, and that

$$\begin{pmatrix} \boldsymbol{x}(k+\beta) \\ \boldsymbol{x}(k+\beta+1) \end{pmatrix}^T \mathbb{R}_{G_0^R} \begin{pmatrix} \boldsymbol{x}(k+\beta) \\ \boldsymbol{x}(k+\beta+1) \end{pmatrix} > 0$$

holds in the domain $\mathcal{D}_R$. Thus, convergence is established for every trajectory that arrives after β samples in $\mathcal{D}_S$, where $\mathcal{D}_S$ is the largest Lyapunov level set contained in $\mathcal{D}_R \bigcap \mathcal{D}$.

The above results can easily be extended using α-sample variation of the Lyapunov function [47].

D. Illustrative examples and discussion

In what follows, we illustrate the conditions on numerical examples. To establish stability, we use sufficient LMI conditions obtained using Lemma 2 on all possible index combinations resulting from Theorem 1. No sample variation has been applied and the LMI conditions have been solved using SeDuMI [48] via the Yalmip [49] Matlab toolbox. Since the domain is determined by the combination of several R matrices, the minimum trace of these matrices is maximized.

Example 3: Recall the system in Example 1, for which the true domain of attraction is given by $\mathcal{D}_S = \{(x_1, x_2) \| x_1| < 1\}$ and consider a TS model constructed for $a = 2$. In this case, using classical conditions, it is not possible to establish (local) stability of the model. However, a feasible solution is obtained using the conditions of Theorem 1, with $G_0^P = G_0^R = \emptyset$, $G_0^N = G_0^M = \{0\}$, R having the structure $R = \mathrm{diag}(R_{11},\, 0,\, R_{33},\, 0)$. The results are illustrated in Figure 1. In this figure, the red o markers denote those points that are in $\mathcal{D}_R$ and the solid lines represent the Lyapunov level sets. As can be seen, the whole domain $\mathcal{D}_S = \{(x_1, x_2) \| x_1| < 1\}$ is recovered (up to numerical errors). $\square$

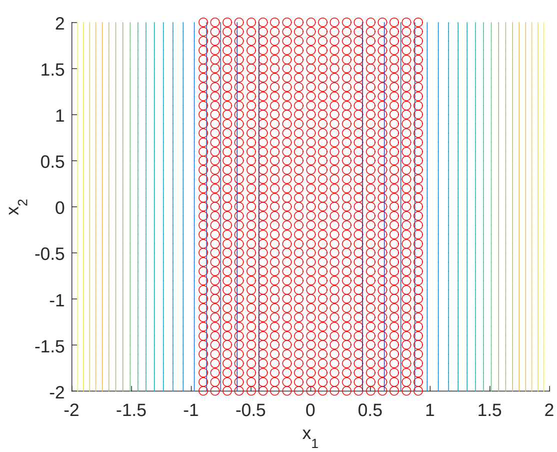


Fig. 1. Results for Example 3.

Example 4: Consider the nonlinear system:

$$\begin{aligned} x_1(k+1) &= 3\sin(x_1(k))\exp(x_1(k))x_1(k) \\ x_2(k+1) &= -x_1(k) + x_2^2(k), \end{aligned} \tag{15}$$

with $x_1(k), x_2(k) \in [-2,\, 2]$.

The equivalent TS model is

$$\boldsymbol{x}(k+1) = \sum_{i=1}^{4} h_i(\boldsymbol{z}(k)) A_i \boldsymbol{x}(k),$$

with

$$\begin{aligned}
A_{1,2} &= \begin{pmatrix} 3\sin(-\frac{\pi}{4})\exp(-\frac{\pi}{4}) & 0 \\ -1 & \pm 2 \end{pmatrix}, \\
A_{3,4} &= \begin{pmatrix} 3\sin(2)\exp(2) & 0 \\ -1 & \pm 2 \end{pmatrix}, \\
w_{1,1} &= \frac{\sin(2)\exp(2) - \sin(x_1(k))\exp(x_1(k))}{\sin(2)\exp(2) - \sin(-\frac{\pi}{4})\exp(-\frac{\pi}{4})}, \\
w_{1,2} &= 1 - w_{1,1}, \\
w_{2,1} &= \frac{2 - x_2(k)}{4}, \quad w_{2,2} = 1 - w_{2,1}, \\
h_1 &= w_{1,1}w_{2,1}, \quad h_2 = w_{1,1}w_{2,2}, \\
h_3 &= w_{1,2}w_{2,1}, \quad h_4 = w_{1,2}w_{2,2}.
\end{aligned}$$

The origin is locally asymptotically stable, with the region of attraction given by $x_1 \in [-0.35, 0.25]$ and $x_2 \in (-1, 1)$. However, the (local) stability of the TS model cannot be proven using classical results. Furthermore, no domain can be recovered using a quadratic Lyapunov function; and the approach in [39] results in a domain of attraction reduced to the equilibrium point, thus it fails.

In what follows, we consider a domain $\mathcal{D}_R$ with $\mathbb{R}_{G_0^R}$ having the form $\mathbb{R}_{G_0^R} = \begin{pmatrix} (o) & 0 \\ 0 & (o) \end{pmatrix}$, where the (o) denotes values to be computed. This form – if not taking into account the indices in G_0^R – gives a direct relation between consecutive samples. We start with the simplest case, $G_0^P = \{0\}$, $G_0^M = \{0\}$, $G_0^R = \{0, 0, 1\}$, and include more and more past samples in the membership functions. The results are graphically illustrated in Figure 2. In these figures, the continuous lines represent the Lyapunov level sets and $*$ denote those points that are in $\mathcal{D}_R$. The stability domain $\mathcal{D}_S$ is given by the largest Lyapunov level set included in $\mathcal{D}_R$.

The following results have been obtained:

R_1: $G_0^P = \{0\}$, $G_0^M = \{0\}$, $G_0^R = \{0, 0, 1\}$: as can be seen in Figure 2(a), the domain is reduced to zero, as there is no Lyapunov level set included in $\mathcal{D}_R$. Regarding the size of the LMI problem before relaxations, the number of decision variables is $N_d = 684$, and the number of lines is $N_l = 256$. A small domain can be recovered using a past sample in the Lyapunov function.

R_2: $G_0^P = \{0\}$, $G_0^M = \{0\}$, $G_0^R = \{-1, 0, 0, 1\}$: in order to increase the domain, we use a delayed index in R. As can be seen in Figure 2(b), the domain is considerably increased. The size of the problem is $N_d = 2604$, $N_l = 1024$.

R_3: $G_0^P = \{-1, 0\}$, $G_0^M = \{-1, 0\}$, $G_0^R = \{-2, -1, 0, 0, 1\}$: the domain is again increased, as illustrated in Figure 2(c), $N_d = 10416$, $N_l = 4096$. For comparison, here we have also drawn the largest Lyapunov level set obtained for R_2.

As illustrated in Figure 2, the choice of the indices G_0^R is more important than the choice of indices in the Lyapunov function. $\square$

It should be noted that when the membership functions depend on past samples, to draw the domain corresponding to the current sample one should take into account all past samples that lead to the current one. In order to avoid this exhaustive search and to illustrate the domain $\mathcal{D}_R$ and the level sets, the predicted level sets and domain are represented, as resulting from the Lyapunov-like function $V(\boldsymbol{x}) = \boldsymbol{x}(k+\beta)^T \mathbb{P}_{G_0^P} \boldsymbol{x}(k+\beta)$, with $\beta = 1$ (R_2) and $\beta = 2$ (R_3). It can be seen that by increasing the number of delays used, the recovered region is increased. However, it has to be kept in mind that generally the implicit equation that defined $\mathcal{D}_R$ is very hard to solve.

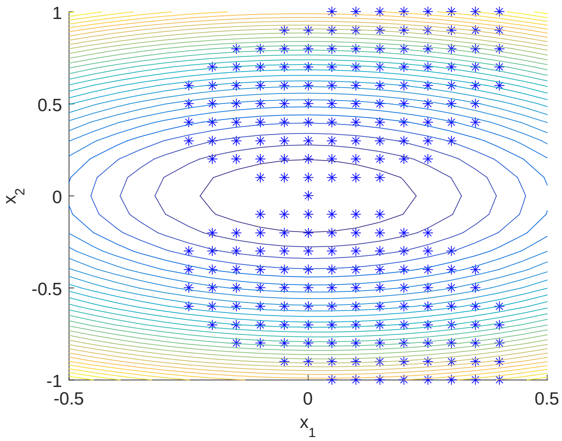
Note that although we conjectured in our previous paper [39] that one may represent predicted Lyapunov level sets, the rigorous mathematical proofs, as presented in Section III-C have been missing. In fact this result enables us to determine a large domain of attraction even if the current Lyapunov level sets lead to a small domain.

It should also be noted that, in general, using an unstructured/ full $\mathbb{R}_{G_0^R}$, although it offers extra degrees of freedom through the larger number of decision variables compared to e.g., a block-diagonal one, may lead to a smaller region of attraction. This is because taking a full $\mathbb{R}_{G_0^R}$ allows the solver to set the local region $\mathcal{D}_R$ at its best convenience to fulfill the conditions of Theorem 1.

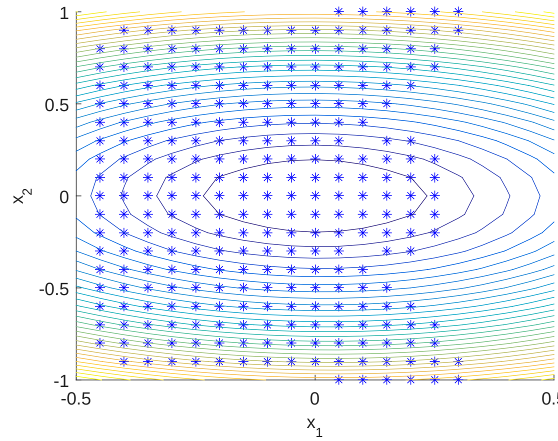
Based on the simulation results, we conjecture that for an appropriate structure of the R_i matrices, with the introduction of further delays, the estimated domain of attraction will asymptotically converge to the true domain of attraction. However, the proof of this conjecture and the choice of the appropriate structure of R_i is left for future research.

IV. Local stabilization

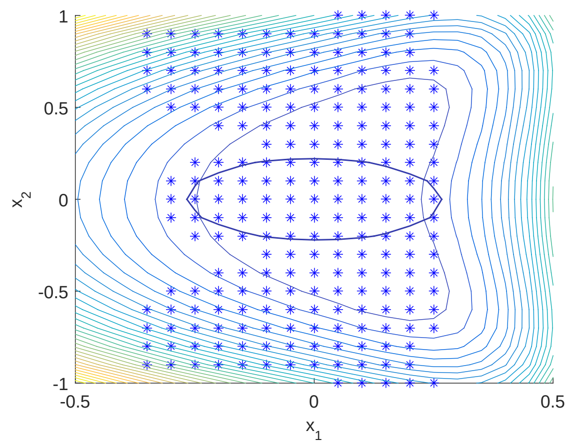
In what follows, we will extend the results from the previous section to local stabilization.



(a) Result for R_1



(b) Result for R_2



(c) Result for R_3. The ellipsis illustrates the stability domain obtained for R_2.

Fig. 2. Results for Example 4.

A. Controller design conditions

In this section, consider the controller design problem for the system

$$\boldsymbol{x}(k+1) = \mathbb{A}_{G_0^A}\boldsymbol{x}(k) + \mathbb{B}_{G_0^B}\boldsymbol{u}(k), \tag{16}$$

using a controller of the form

$$\boldsymbol{u}(k) = -\mathbb{F}_{G_0^F}\mathbb{H}_{G_0^H}^{-1}\boldsymbol{x}(k). \tag{17}$$

The closed-loop system can be expressed as

$$\boldsymbol{x}(k+1) = (\mathbb{A}_{G_0^A} - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F}\mathbb{H}_{G_0^H}^{-1})\boldsymbol{x}(k). \tag{18}$$

Since we use a specific controller structure and develop sufficient conditions, we do not make any assumption on the stabilizability/controllability of the equilibrium point $\boldsymbol{x} = 0$ of the closed-loop system. If the system is not controllable, then, independent of the Lyapunov function, controller gains and relaxations used, the domain $\mathcal{D}_R$ will be reduced to the equilibrium point.

In what follows, two Lyapunov functions will be considered:

- Case 1: $V(\boldsymbol{x}(k)) = \boldsymbol{x}^T(k)\mathbb{H}_{G_0^H}^{-T}\mathbb{P}_{G_0^P}\mathbb{H}_{G_0^H}^{-1}\boldsymbol{x}(k)$ and
- Case 2: $V(\boldsymbol{x}(k)) = \boldsymbol{x}^T(k)\mathbb{P}_{G_0^P}^{-1}\boldsymbol{x}(k)$ and

These two Lyapunov functions have frequently been used in the literature. Although the second one $V(\boldsymbol{x}(k)) = \boldsymbol{x}^T(k)\mathbb{P}_{G_0^P}^{-1}\boldsymbol{x}(k)$ is a special case – choice of $\mathbb{H}_{G_0^H} = \mathbb{P}_{G_0^P}$ – of the first one $V(\boldsymbol{x}(k)) = \boldsymbol{x}^T(k)\mathbb{H}_{G_0^H}^{-T}\mathbb{P}_{G_0^P}\mathbb{H}_{G_0^H}^{-1}\boldsymbol{x}(k)$, since the final LMI conditions are obtained in different ways for the two functions, the results obtained with the first one will not always include the second one. Depending on the problem considered, one or the other may be used.

Let us first consider the Lyapunov function in the Case 1 above, together with a domain $\mathcal{D}_R$ defined as

$$\begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}^T \begin{pmatrix} \mathbb{H}_{G_0^H}^{-T} & 0 \\ 0 & \mathbb{H}_{G_1^H}^{-T} \end{pmatrix} \mathbb{R}_{G_0^R} (*) > 0. \tag{19}$$

The following result can be established:

Theorem 2: The closed-loop system (18) is locally asymptotically stable in the domain $\mathcal{D}_S$, if there exist matrices $P_{\boldsymbol{i}_j^P} = P^T_{\boldsymbol{i}_j^P}$, $\boldsymbol{i}_j^P = \mathrm{pr}^{\boldsymbol{i}}_{G_j^P}$, $H_{\boldsymbol{i}_j^H}$, $\boldsymbol{i}_j^H = \mathrm{pr}^{\boldsymbol{i}}_{G_j^H}$, and $F_{\boldsymbol{i}_j^F}$, $\boldsymbol{i}_j^F = \mathrm{pr}^{\boldsymbol{i}}_{G_j^F}$, $\boldsymbol{i} \in \mathbb{I}_{G_V}$, $j = 0, 1$, and $R_{\boldsymbol{i}_0^R} = R^T_{\boldsymbol{i}_0^R}$, $\boldsymbol{i}_0^R = \mathrm{pr}^{\boldsymbol{i}}_{G_0^R}$, where $G_V = G_0^P \cup G_1^P \cup (G_0^M \oplus G_0^A) \cup (G_0^F \oplus G_0^B) \cup G_0^R$ so that

$$\begin{pmatrix} -\mathbb{P}_{G_0^P} & (*) \\ \mathbb{A}_{G_0^A}\mathbb{H}_{G_0^H} - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F} & \begin{pmatrix} -\mathbb{H}_{G_1^H} - \mathbb{H}_{G_1^H}^T \\ +\mathbb{P}_{G_1^P} \end{pmatrix} \end{pmatrix} + \mathbb{R}_{G_0^R} < 0$$

holds. Moreover, the region of attraction includes $\mathcal{D}_S$, where $\mathcal{D}_S$ is the largest Lyapunov level set included in $\mathcal{D}_R$.

Proof 2: The difference in the Lyapunov function is

$$\Delta V = \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}^T \begin{pmatrix} -\mathbb{H}_{G_0^H}^{-T}\mathbb{P}_{G_0^P}\mathbb{H}_{G_0^H}^{-1} & 0 \\ 0 & \mathbb{H}_{G_1^H}^{-T}\mathbb{P}_{G_1^P}\mathbb{H}_{G_1^H}^{-1} \end{pmatrix} (*)$$

and the closed-loop system can be expressed as

$$\begin{pmatrix} \mathbb{A}_{G_0^A} - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F}\mathbb{H}_{G_0^H}^{-1} & -I \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix} = 0.$$

Furthermore, in the domain $\mathcal{D}_R$, (19) holds and the S-procedure can be applied. Thus, in this domain, (18) is locally asymptotically stable, if there exists $\mathcal{M}$ so that

$$\begin{aligned} &\begin{pmatrix} -\mathbb{H}_{G_0^H}^{-T}\mathbb{P}_{G_0^P}\mathbb{H}_{G_0^H}^{-1} & 0 \\ 0 & \mathbb{H}_{G_1^H}^{-T}\mathbb{P}_{G_1^P}\mathbb{H}_{G_1^H}^{-1} \end{pmatrix} \\ &+ \mathcal{M}\begin{pmatrix} \mathbb{A}_{G_0^A} - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F}\mathbb{H}_{G_0^H}^{-1} & -I \end{pmatrix} + (*) \\ &+ \begin{pmatrix} \mathbb{H}_{G_0^H}^{-T} & 0 \\ 0 & \mathbb{H}_{G_1^H}^{-T} \end{pmatrix} \mathbb{R}_{G_0^R} \begin{pmatrix} \mathbb{H}_{G_0^H}^{-1} & 0 \\ 0 & \mathbb{H}_{G_1^H}^{-1} \end{pmatrix} < 0. \end{aligned}$$

Choosing $\mathcal{M} = \begin{pmatrix} 0 \\ \mathbb{H}_{G_1^H}^{-T} \end{pmatrix}$ and congruence with $\begin{pmatrix} \mathbb{H}_{G_0^H}^T & 0 \\ 0 & \mathbb{H}_{G_1^H}^T \end{pmatrix}$ leads to the conditions of Theorem 2. Furthermore, since the condition (19) holds only in the domain $\mathcal{D}_R$, the region of attraction includes the largest Lyapunov level set contained in $\mathcal{D}_R$. □

For the Lyapunov function in Case 2, consider a domain $\mathcal{D}_R$ defined as

$$\begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}^T \begin{pmatrix} \mathbb{H}_{G_0^H}^{-T} & 0 \\ 0 & \mathbb{P}_{G_1^P}^{-1} \end{pmatrix} \mathbb{R}_{G_0^R} \begin{pmatrix} \mathbb{H}_{G_0^H}^{-1} & 0 \\ 0 & \mathbb{P}_{G_1^P}^{-1} \end{pmatrix} (*) > 0 \tag{20}$$

For this case, the following result can be established:

Theorem 3: The closed-loop system (18) is locally asymptotically stable in the domain $\mathcal{D}_S$, if there exist matrices $P_{\boldsymbol{i}_j^P} = P^T_{\boldsymbol{i}_j^P}$, $\boldsymbol{i}_j^P = \mathrm{pr}^{\boldsymbol{i}}_{G_j^P}$, $H_{\boldsymbol{i}_j^H}$, $\boldsymbol{i}_j^H = \mathrm{pr}^{\boldsymbol{i}}_{G_j^H}$, and $F_{\boldsymbol{i}_j^F}$, $\boldsymbol{i}_j^F = \mathrm{pr}^{\boldsymbol{i}}_{G_j^F}$, $\boldsymbol{i} \in \mathbb{I}_{G_V}$, $j = 0, 1$, and $R_{\boldsymbol{i}_0^R} = R^T_{\boldsymbol{i}_0^R}$, $\boldsymbol{i}_0^R = \mathrm{pr}^{\boldsymbol{i}}_{G_0^R}$, where $G_V = G_0^P \cup G_1^P \cup (G_0^M \oplus G_0^A) \cup (G_0^F \oplus G_0^B) \cup G_0^R$ so that

$$\begin{pmatrix} -\mathbb{H}_{G_0^H} - \mathbb{H}_{G_0^H}^T + \mathbb{P}_{G_0^P} & (*) \\ \mathbb{A}_{G_0^A}\mathbb{H}_{G_0^H} - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F} & -\mathbb{P}_{G_1^P} \end{pmatrix} + \mathbb{R}_{G_0^R} < 0$$

holds. Moreover, the region of attraction includes $\mathcal{D}_S$, where $\mathcal{D}_S$ is the largest Lyapunov level set included in $\mathcal{D}_R$.

Proof 3: Writing the difference in the Lyapunov function, considering the domain (20), using Lemma 1 with $\mathcal{M} = \begin{pmatrix} 0 \\ \mathbb{P}_{G_1^P}^{-1} \end{pmatrix}$ and congruence with $\begin{pmatrix} \mathbb{H}_{G_0^H}^T & 0 \\ 0 & \mathbb{P}_{G_1^P} \end{pmatrix}$ leads to

$$\begin{pmatrix} -\mathbb{H}_{G_0^H}^T\mathbb{P}_{G_0^P}^{-1}\mathbb{H}_{G_0^H} & (*) \\ \mathbb{A}_{G_0^A}\mathbb{H}_{G_0^H} - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F} & -\mathbb{P}_{G_1^P} \end{pmatrix} + \mathbb{R}_{G_0^R} < 0.$$

Applying Proposition 1 gives the condition of Theorem 3. Furthermore, since the condition (20) holds only in the domain $\mathcal{D}_R$, the region of attraction includes the largest Lyapunov level set contained in $\mathcal{D}_R$. $\square$

B. Complexity analysis

Let us now discuss the choice of the delays in G_0^P, G_0^H, G_0^F, and G_0^R: G_0^R should again contain all the appearing indices; a reasonable choice is $G_0^H = G_0^F$, since they appear multiplied by A_z and B_z, respectively. Thus, G_V is now reduced to $G_V = G_0^P \cup G_1^P \cup (G_0^H \oplus G_0^A)$. Taking into account that G_0^H cannot contain positive indices (as they refer to future states) and following the reasoning in [14], from the point of conservatism reduction approaches and reduced computational complexity, for Case 1, the best choices are $G_0^P = \{0, 0, \ldots, 0\}$ and $G_0^F = G_0^H = \{0, 0, \ldots, 0\}$ and for Case 2, $G_0^P = \{-1, -1, \ldots, -1\}$ and $G_0^F = G_0^H = \{0, 0, \ldots, 0, -1, \ldots, -1\}$. These choices reduce the number of sums as follows

- Case 1: assuming $|G_0^P| = |G_0^H| = n_P$, the number of sums in Theorem 2 is $2n_P + 1$ and the number of LMIs to be solved (before relaxations) is r^{2n_p+1}.
- Case 2: assuming $|G_0^P| = |G_0^H| = 2n_P$, the number of sums in Theorem 3 is $2n_P + 1$ and the number of LMIs to be solved (before relaxations) is r^{2n_p+1}.

Similar to the stability analysis, under the assumption that $G_0^F = G_0^H$ and all the indices are included in G_0^R, for both cases, the number of decision variables is $N_d = 2r^{|G_0^H|} \times n^2 + r^{|G_0^P|} \times \frac{n(n+1)}{2} + r^{|G_0^R|} \times n(2n+1)$ and the number of rows in the LMI problem is $N_l = 2r^{|G_0^R|}n$. It is possible to consider a Lyapunov-like function $V(\boldsymbol{x}) = \boldsymbol{x}(k+\beta)^T \mathbb{P}_{G_0^P} \boldsymbol{x}(k+\beta)$, as in the case of stability analysis.

C. Examples and discussion

In this section we illustrate the controller design results on two examples. We use sufficient LMI conditions obtained using Lemma 2 on all possible index combinations, and solve them using SeDuMI [48] via the Yalmip [49] Matlab toolbox. For the domain, block-diagonal R matrices are considered, and the minimum of the traces of all the R matrices is maximized.

Example 5: Consider the nonlinear system:

$$\begin{aligned} x_1(k+1) &= 4\frac{x_1^2(k)}{1+x_1^2(k)} + 0.1x_2(k) \\ x_2(k+1) &= -x_1(k) - \frac{1}{2}x_2(k) + u(k). \end{aligned} \tag{21}$$

The equivalent TS model is

$$\boldsymbol{x}(k+1) = \sum_{i=1}^{2} h_i(\boldsymbol{z}(k)) \left(A_i \boldsymbol{x}(k) + Bu(k)\right)$$

with

$$A_1 = \begin{pmatrix} -2 & 0.1 \\ -1 & -\frac{1}{2} \end{pmatrix}, \quad A_2 = \begin{pmatrix} 2 & 0.1 \\ -1 & -\frac{1}{2} \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$
$$h_1(k) = \frac{1}{2} - \frac{x_1(k)}{1+x_1^2(k)}, \quad h_2(k) = 1 - h_1(k).$$

Both local models are unstable and controllable. Controller design using either quadratic or nonquadratic Lyapunov functions results in unfeasible LMIs; and the approach in [40] gives the domain of attraction $\mathcal{D}_R = \{0\}$, i.e., only the equilibrium point.

To illustrate local stabilization, consider the conditions of Theorem 2. For the different choices of the delays, the results are presented in Figure 3. In all the figures, the $*$ denotes the domain $\mathcal{D}_R$, the Lyapunov level sets are drawn, and the green o denote initial values from which trajectories converge to zero.

To circumvent the issue of the graphical representation with past samples, the predicted domain and/or level sets are illustrated, as discussed in Section III-D.

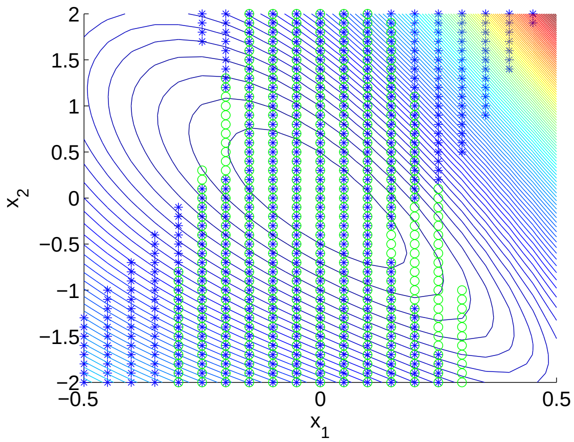
R_1: $G_0^P = \{0, 0, 0\}$, $G_0^H = G_0^F = \{0, 0\}$, $G_0^R = \{0, 0, 0, 1, 1, 1\}$: as can be seen in Figure 3(a), a small domain is obtained. The domain can be somewhat increased by using more indices. The size of the LMI problem is $N_d = 33088$ and $N_l = 4096$. A smaller number of indices results in $\mathcal{D}_R = \{0\}$.

Simulations indicate that including past samples in the Lyapunov function does not have any positive effect, but the domain increases when including past samples in its definition.

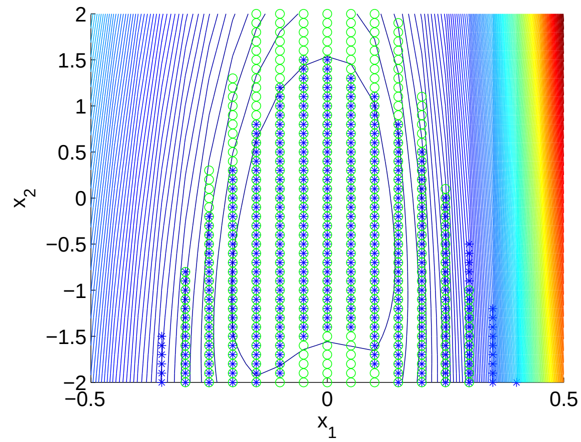
R_2: $G_0^P = \{0, 0, 0\}$, $G_0^H = G_0^F = \{0, 0\}$, $G_0^R = \{-1, 0, 0, 0, 1, 1, 1\}$: the results are shown in Figure 3(b). The size of the LMI problem is $N_d = 131392$ and $N_l = 65536$.

R_3: $G_0^P = \{0, 0, 0\}$, $G_0^H = G_0^F = \{0, 0\}$, $G_0^R = \{-2, -1, 0, 0, 0, 1, 1, 1\}$: the size of the LMI problem is $N_d = 524608$, $N_l = 262144$, but, as can be seen in Figure 3(c), the domain further increases.

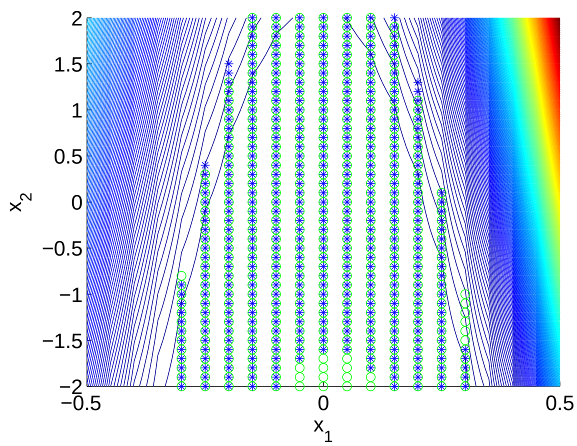
The domain can be further increased by using more delays in the domain matrix. Moreover, simulations indicate that the predicted Lyapunov level sets asymptotically converge to the domain where the system is stabilized. $\square$



(a) Result for R_1



(b) Result for R_2



(c) Result for R_3

Fig. 3. Results for Example 5

It has to be noted that using delays in $\mathbb{H}_{G_0^H}$ and $\mathbb{F}_{G_0^F}$ requires the specification of the initial conditions $\boldsymbol{x}(-1)$, $\boldsymbol{x}(-2)$, etc., and the controller will need to store or to have access to past states. This is in particular the case of Theorem 3, where, in order to obtain less conservative results, G_0^F and G_0^H should include -1. However, this is not a significant shortcoming, as any current microcontroller is able to do it.

Similar to stability analysis, a structured $\mathbb{R}_{G_0^R}$ seems to lead to an increase of the region. However, one should keep in mind that the region is actually determined not only by $\mathbb{R}_{G_0^R}$, but also by $\mathbb{H}_{G_0^H}$ and $\mathbb{R}_{G_0^R}$. The elucidation of the problem of how exactly the structure of $\mathbb{R}_{G_0^R}$ should be chosen such that the region is optimized is left for future research.

Finally, let us consider a more realistic example of designing a local stabilizing controller for an inverted pendulum on a cart.

Example 6: The following continuous-time model of an inverted pendulum on a cart (see Section 2.6 in [2]) is considered:

$$\begin{aligned}\dot{x}_1 =& x_2\\ \dot{x}_2 =& \frac{g\sin(x_1) - amlx_2^2\sin(x_1)\cos(x_1) - a\cos(x_1)u}{\frac{4l}{3} - aml\cos^2(x_1)},\end{aligned} \tag{22}$$

where x_1 is the angle of the pendulum, and x_2 is the angular velocity, u is the force applied to the cart. The model parameters and their values can be found in Table I; $a = \frac{1}{m+M}$.

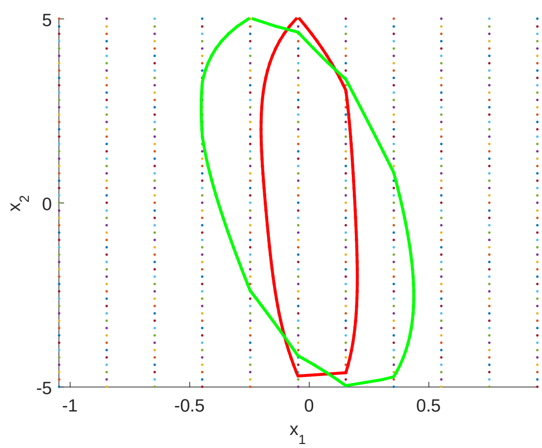
The system is discretized using a forward Euler discretization with the sampling time $T_s = 0.01$ and is rewritten as

$$\begin{aligned}x_1(k+1) &= x_1(k) + T_s x_2(k)\\ x_2(k+1) &= T_s\frac{g - amlx_2(k)^2\cos(x_1(k))}{4l/3 - aml\cos(x_1(k))^2}\frac{\sin(x_1(k))}{x_1(k)}x_1(k)\\ &\quad + x_2(k) + T_s\frac{-a\cos(x_1(k))}{4l/3 - aml\cos(x_1(k))^2}u(k).\end{aligned}$$

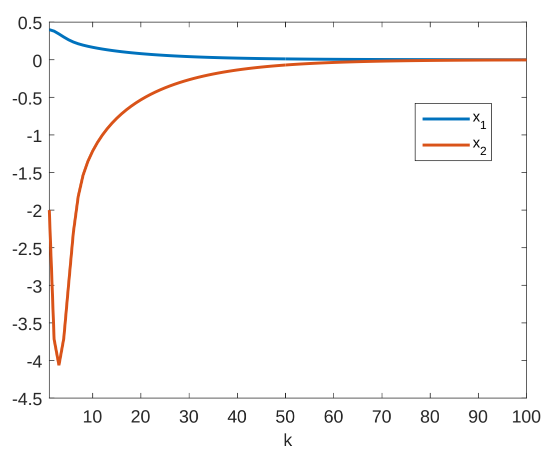
TABLE I
PARAMETER TABLE

Notation	Value	Description
g $[m^s/s]$	9.8	gravitational acceleration
m [kg]	0.2	mass of pendulum
M [kg]	1.61	mass of cart
γ [N/rad/s]	0.4898	friction coefficient
l [m]	0.335	half-length of pendulum
J $[kg\,m^2]$	0.0232	moment of inertia
K_m [-]	6.5914	PWM gain
σ [rad/s]	4.8	max angular velocity

The physical limitations are $x_1 \in [-\frac{\pi}{3}, \frac{\pi}{3}]$ and $x_2 \in [-5, 5]$. We use the sector nonlinearity approach to construct an equivalent TS model in this domain: there are two nonlinearities, $\frac{g-amlx_2(k)^2\cos(x_1(k))}{4l/3-aml\cos(x_1(k))^2}\frac{\sin(x_1(k))}{x_1(k)} \in [17, 24]$, and $\frac{-a\cos(x_1(k))}{4l/3-aml\cos(x_1(k))^2} \in [-1.4, -0.6]$, leading to a four-rule TS model. To illustrate the improvement in the domain of attraction we consider the control law $u(k) = -F_z H_z^{-1}\boldsymbol{x}(k)$, i.e., $G_0^F = G_0^H = \{0\}$, a single-sum nonquadratic Lyapunov function, i.e., $G_0^P = \{0\}$, and $G_0^R = \{0\}$. The largest Lyapunov level set obtained by simply solving the feasibility problem for controller design, without taking into account the domain is drawn with red line in Figure 4(a). The largest Lyapunov level set obtained with our approach, with the same settings and considering $G_0^R = \{0\}$ and R_i diagonal matrices is given by the green line in Figure 4(a). In the same figure, the $\cdot$ (dots) denote values that are in the domain $\mathcal{D}_R$, and, as can be seen, $\mathcal{D}_R$ covers the whole modeling domain $\mathcal{D}$. A simulation of the closed-loop system from the initial value $\boldsymbol{x}(0) = [0.4, -2]^T$ is illustrated in Figure 4(b): the controller stabilizes the system.



(a) Lyapunov level sets for Example 6, $G_0^P = \{0\}$.



(b) Trajectory of the closed-loop system.

Fig. 4. Results for Example 6

For this particular example, with the indices chosen as above, including more indices in the definition of the domain matrix does not provide a significant increase of the domain of attraction.

Next, we tested the domain of attraction that can be obtained with $G_0^F = G_0^H = \{0, 0\}$, $G_0^P = \{0, 0, 0\}$ and $G_0^R = \{0, 0, 0, 1, 1, 1\}$, which matches all the indices in the conditions. The obtained results can be seen in Figure 5 – green - our approach, red - feasibility only. Similarly to the previous case, the inclusion of more indices in the domain matrix does not lead to a significant increase of the domain of attraction. $\square$

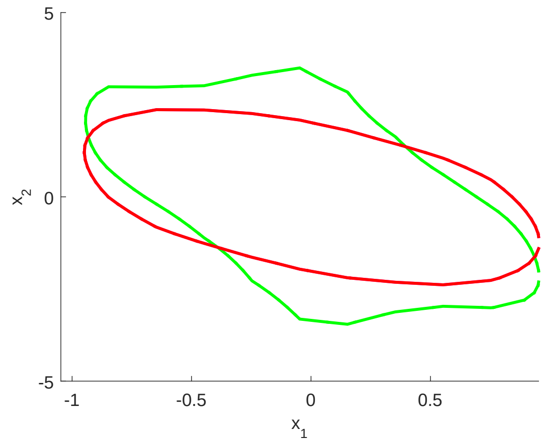


Fig. 5. Lyapunov level sets for Example 6, $G_0^P = \{0, 0, 0\}$.

As illustrated by the examples, choosing the structure of the domain matrix and the indices to be used is crucial in obtaining a large domain of attraction. Furthermore, the correct choice is a complex and application-dependent task. The determination of a general rule is currently left for future research.

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