Analysis and design for continuous-time string-connected Takagi-Sugeno systems

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Abstract

Distributed systems consist of interconnected, lower-dimensional subsystems. For such systems, distributed analysis and design present several advantages, such as modularity, easier analysis and design, and reduced computational complexity. A special case of distributed systems is when the subsystems are connected in a string. Applications include distributed process control, traffic and communication networks, irrigation systems, hydropower valleys, etc. By exploiting such a structure, in this paper, we propose conditions for the distributed stability analysis of Takagi-Sugeno fuzzy systems connected in a string. These conditions are also extended to observer and controller design and illustrated on numerical examples.

Keywords: string-connected systems, distributed systems, TS models, local stability

1. Introduction

Takagi-Sugeno (TS) fuzzy systems [1] are nonlinear, convex combinations of local linear models, that can exactly represent a large class of nonlinear systems.

For the stability analysis and observer and controller design for TS models, Lyapunov's direct approach can be used, employing common quadratic [2, 3], piecewise quadratic [4] or, recently, non-quadratic [5, 6, 7, 8] Lyapunov functions. Based on these Lyapunov functions, the stability or design conditions are generally derived in the form of linear matrix inequalities (LMIs).

Although a real improvement in the design conditions for discrete-time systems [5, 9, 10, 11] has been brought by the use of nonquadratic Lyapunov functions, such Lyapunov functions have also been used for continuous-time TS models [12, 13, 14, 7]. To reduce the conservativeness of the conditions, properties of the membership functions have been introduced [15], or the complexity of the LMI was reduced [16, 17].

Many physical systems, such as power systems, communication networks, economic systems, and traffic and communication networks, irrigation systems, hydropower valleys, etc., are composed of interconnections of lower-dimensional subsystems. Recently, decentralized analysis and control design for such systems has received much attention [18, 19, 20, 21, 22, 23].

Stability analysis of distributed TS systems mainly relies on the existence of a common quadratic Lyapunov function for each subsystem [24, 25, 22]. Most results make use of the assumption that the number of subsystems and some bounds on the interconnection terms are known a priori, and the analysis of the subsystems is performed in parallel. For instance, an early result has been formulated in [24] and relaxed in [26]. In these approaches, LMI conditions for establishing the stability of the individual subsystems are solved in parallel, and afterward the stability of the whole system is verified. For hybrid linear-fuzzy systems, a similar method for establishing the stability of the distributed system has been proposed in [25]. For distributed TS systems with affine consequents, but linear interconnection terms among the subsystems, an approach based on piecewise Lyapunov functions has been developed in [22]. Stability analysis of uncertain distributed TS systems has been investigated e.g., in [19].

All the above mentioned results assume that any two subsystems in the distributed system may be interconnected. While this assumption makes the results generally applicable, it also introduces conservativeness. In this paper we develop conditions for the stability analysis of string-connected TS systems, i.e., distributed TS systems in which each subsystem is connected only to its two neighbors. The coupling between the subsystems is realized through their states. The approach is also extended to controller and observer design. To our best knowledge, analysis and design of this specific structure have not been addressed in the literature. By exploiting the structure, less conservative conditions are obtained. Moreover, to develop the conditions, we employ nonquadratic Lyapunov functions.

The structure of the paper is as follows. Section 2 presents the general form of the TS models and the notations used in this paper. Section 3 describes the proposed stability conditions. Sections 4 and 5 present the conditions for observer and controller design, respectively, and illustrate them on numerical examples. Finally, Section 6 concludes the paper.

2. Preliminaries

A centralized TS fuzzy system is of the form[1]

$$\begin{aligned}\dot{\boldsymbol{x}} &= \sum_{i=1}^{m} w_i(\boldsymbol{z})(A_i\boldsymbol{x} + B_i\boldsymbol{u})\\ \boldsymbol{y} &= \sum_{i=1}^{m} w_i(\boldsymbol{z})C_i\boldsymbol{x}\end{aligned}$$

where $\boldsymbol{x} \in \mathbb{R}^{n_x}$ is the vector of the state variables, $\boldsymbol{u} \in \mathbb{R}^{n_u}$ is the input vector, $\boldsymbol{y} \in \mathbb{R}^{n_y}$ is the measurement vector. In the equations above, $A_i \in \mathbb{R}^{n_x \times n_x}$, $B_i \in \mathbb{R}^{n_x \times n_u}$, and $C_i \in \mathbb{R}^{n_y \times n_x}$, $i = 1, 2, \ldots, m$ represent the matrices of the ith local linear model and w_i, $i = 1, 2, \ldots, m$

[1]Since all the variables in the system description are time-varying, for the ease of notation, we do not explicitly denote the time.

are the corresponding membership functions, which depend on the scheduling variables $\boldsymbol{z}$. The scheduling variables generally may depend on the states, inputs, or other exogenous variables. In order to avoid the need of solving implicit equations, for the controller design, it is assumed that the scheduling variables do not depend on the input. The membership functions $w_i(\boldsymbol{z})$ are assumed to be normalized, i.e., $w_i(\boldsymbol{z}) \in [0, 1]$ and $\sum_{i=1}^{m} w_i(\boldsymbol{z}) = 1$, $\forall \boldsymbol{z}$. In what follows, for simplicity, we will use the notation $X_z = \sum_{i=1}^{m} w_i(\boldsymbol{z}) X_i$, $X_z^{-1} = (\sum_{i=1}^{m} w_i(\boldsymbol{z}) X_i)^{-1}$.

In this paper we focus on distributed systems where the subsystems are connected in a bidirectional string, as shown in Figure 1.

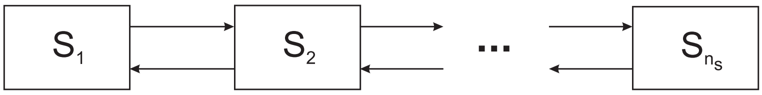

Figure 1: Subsystems connected in a string.

Such interconnections are common e.g., in flow processes or production processes. The n_s subsystems are described by the TS models

$$
\begin{aligned}
\dot{\boldsymbol{x}}_l &= \sum_{i=1}^{m_l} w_i^l(\boldsymbol{z}_l)(A_i^l \boldsymbol{x}_l + A_i^{l,l-1}\boldsymbol{x}_{l-1} + A_i^{l,l+1}\boldsymbol{x}_{l+1} + B_i^l \boldsymbol{u}_l) \\
\boldsymbol{y}_l &= \sum_{i=1}^{m_l} w_i^l(\boldsymbol{z}_l) C_i^l \boldsymbol{x}_l
\end{aligned}
$$

or

$$
\begin{aligned}
\dot{\boldsymbol{x}}_l &= A_z^l \boldsymbol{x}_l + A_z^{l,l-1}\boldsymbol{x}_{l-1} + A_z^{l,l+1}\boldsymbol{x}_{l+1} + B_z^l \boldsymbol{u}_l \\
\boldsymbol{y}_l &= C_z^l \boldsymbol{x}_l
\end{aligned}
\tag{1}
$$

for $l = 1, 2, \ldots, n_\mathrm{s}$, with $A_i^{1,0} = 0$, $i = 1, 2, \ldots, m_1$, $\boldsymbol{x}_0 \equiv 0$, and $A_i^{n_\mathrm{s}, n_\mathrm{s}+1} = 0$, $i = 1, 2, \ldots, m_{n_\mathrm{s}}$, $\boldsymbol{x}_{n_\mathrm{s}+1} \equiv 0$, i.e., for the ease of notation the states of the 0th and the $n_\mathrm{s}+1$th subsystems are considered to be identical to 0.

In the above descriptions, $A_i^l \in \mathbb{R}^{n_{x_l} \times n_{x_l}}$, $B_i^l \in \mathbb{R}^{n_{x_l} \times n_{u_l}}$, $C_i^l \in \mathbb{R}^{n_{y_l} \times n_{x_l}}$, $A_i^{l,l-1} \in \mathbb{R}^{n_{x_l} \times n_{x_{l-1}}}$, and $A_i^{l,l+1} \in \mathbb{R}^{n_{x_l} \times n_{x_{l+1}}}$, $i = 1, 2, \ldots, m_l$, i.e., the dimensions of the subsystems may differ from one another.

Throughout the paper it is assumed that the membership functions of each subsystem are normalized, i.e., $w_i^l(\boldsymbol{z}_l) \geq 0$, $\sum_{i=1}^{m} w_i^l(\boldsymbol{z}_l) = 1$, $\forall \boldsymbol{z}_l$, $l = 1, 2, \ldots, n_\mathrm{s}$. Moreover, the matrices I and 0 denote the identity and the zero matrices of the appropriate dimensions, and $(*)$ denotes the term induced by symmetry, i.e., $\begin{pmatrix} A & B \\ (*) & C \end{pmatrix} = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$ and $A + (*) = A + A^T$.

In what follows, we consider first stability analysis, and next observer and controller design for the string-connected TS model. To derive the stability and design conditions, we use nonquadratic Lyapunov functions.

3. Stability analysis

Consider the string-connected, *autonomous* TS fuzzy model composed of n_s subsystems as follows:

$$\dot{\boldsymbol{x}}_l = A_z^l \boldsymbol{x}_l + A_z^{l,l-1}\boldsymbol{x}_{l-1} + A_z^{l,l+1}\boldsymbol{x}_{l+1} \tag{2}$$

for $l = 1, 2, \ldots, n_\mathrm{s}$, with $A_z^{1,0} = 0$, $\boldsymbol{x}_0 \equiv 0$, and $A_z^{n_\mathrm{s},n_\mathrm{s}+1} = 0$, $\boldsymbol{x}_{n_\mathrm{s}+1} \equiv 0$.

3.1. Stability conditions

For the system (2), the following result can be formulated:

Theorem 1. *The string-connected TS system with the subsystems described by* (2) *is locally asymptotically stable if there exist matrices* $P_i^l = (P_i^l)^T > 0$*,* $i = 1, 2, \ldots, m_l$*, and scalars* $d_l > 0$*,* $l = 1, \ldots, n_\mathrm{s}$*, so that*

$$\begin{pmatrix} d_{l-1}P_z^{l-1}A_z^{l-1} + (*) + d_{l-1}\dot{P}_z^{l-1} & X_{l,l-1} \\ (*) & d_l P_z^l A_z^l + (*) + \dot{P}_z^l \end{pmatrix} < 0$$

for $l = 2, \ldots, n_\mathrm{s} - 1$*, where* $X_{l,l-1} = 2d_l (A_z^{l,l-1})^T P_z^l + 2d_{l-1}P_z^{l-1}A_z^{l-1,l}$.

Proof. *Following the lines of the proof in [27], consider the composite nonquadratic Lyapunov function* $V = \sum_{l=1}^{n_\mathrm{s}} 2d_l \boldsymbol{x}_l^T P_z^l \boldsymbol{x}_l$*,* $P_z^l = \sum_{i=1}^{m_l} w_i(\boldsymbol{z})P_i^l$ *with* $P_i^l = P_i^{lT} > 0$*,* $i = 1, 2, \ldots, m_l$*, and* $d_l > 0$*,* $l = 1, \ldots, n_\mathrm{s}$*. The derivative* $\dot{V}$ *can be written as*

$$\begin{aligned}
\dot{V} &= \sum_{l=2}^{n_\mathrm{s}-1} 2d_l \boldsymbol{x}_l^T P_z^l (A_z^l \boldsymbol{x}_l + A_z^{l,l-1}\boldsymbol{x}_{l-1} + A_z^{l,l+1}\boldsymbol{x}_{l+1}) + (*) \\
&\quad + 2d_1 \boldsymbol{x}_1^T P_1 (A_z^1 \boldsymbol{x}_1 + A_z^{1,2}\boldsymbol{x}_2) + (*) \\
&\quad + 2d_{n_\mathrm{s}} \boldsymbol{x}_{n_\mathrm{s}}^T P_{n_\mathrm{s}} (A_z^{n_\mathrm{s}} \boldsymbol{x}_{n_\mathrm{s}} + A_z^{n_\mathrm{s},n_\mathrm{s}-1}\boldsymbol{x}_{n_\mathrm{s}-1}) + (*) + \sum_{l=1}^{n_\mathrm{s}} 2d_l \dot{P}_z^l \\
&= \sum_{l=2}^{n_\mathrm{s}-1} 2d_l \begin{pmatrix} \boldsymbol{x}_{l-1} \\ \boldsymbol{x}_l \\ \boldsymbol{x}_{l+1} \end{pmatrix}^T \begin{pmatrix} 0 & (*) & 0 \\ P_z^l A_z^{l,l-1} & P_z^l A_z^l + (*) + \dot{P}_z^l & P_z^l A_z^{l,l+1} \\ 0 & (*) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{x}_{l-1} \\ \boldsymbol{x}_l \\ \boldsymbol{x}_{l+1} \end{pmatrix} \\
&\quad + 2d_1 \begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{pmatrix}^T \begin{pmatrix} P_z^1 A_z^1 + (*) + \dot{P}_z^1 & (*) \\ P_z^1 A_z^{1,2} & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{pmatrix} \\
&\quad + 2d_{n_\mathrm{s}} \begin{pmatrix} \boldsymbol{x}_{n_\mathrm{s}-1} \\ \boldsymbol{x}_{n_\mathrm{s}} \end{pmatrix}^T \begin{pmatrix} 0 & (*) \\ P_z^{n_\mathrm{s}} A_z^{n_\mathrm{s},n_\mathrm{s}-1} & P_z^{n_\mathrm{s}} A_z^{n_\mathrm{s}} + (*) + \dot{P}_z^{n_\mathrm{s}} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}_{n_\mathrm{s}-1} \\ \boldsymbol{x}_{n_\mathrm{s}} \end{pmatrix} \\
&= \sum_{l=2}^{n_\mathrm{s}} \begin{pmatrix} \boldsymbol{x}_{l-1} \\ \boldsymbol{x}_l \end{pmatrix}^T \begin{pmatrix} d_{l-1}P_z^{l-1}A_z^{l-1} + (*) + d_{l-1}\dot{P}_z^{l-1} & X_{l,l-1} \\ (*) & d_l P_l A_z^l + (*) + d_l \dot{P}_z^l \end{pmatrix} \begin{pmatrix} \boldsymbol{x}_{l-1} \\ \boldsymbol{x}_l \end{pmatrix} \\
&\quad + \boldsymbol{x}_1^T (d_1 P_z^1 A_z^1 + (*) + d_1 \dot{P}_z^1)\boldsymbol{x}_1 + \boldsymbol{x}_{n_\mathrm{s}}^T (d_{n_\mathrm{s}} P_{n_\mathrm{s}} A_z^{n_\mathrm{s}} + (*) + d_{n_\mathrm{s}} \dot{P}_z^{n_\mathrm{s}})\boldsymbol{x}_{n_\mathrm{s}}
\end{aligned}$$

with $X_{l,l-1} = 2d_l(A_z^{l,l-1})^T P_z^l + 2d_{l-1}P_z^{l-1}A_z^{l-1,l}$. *Define*

$$\Gamma_l = \begin{pmatrix} d_{l-1}P_z^{l-1}A_z^{l-1} + (*) + d_{l-1}\dot{P}_z^{l-1} & X_{l,l-1} \\ (*) & d_l P_z^l A_z^l + (*) + d_l \dot{P}_z^l \end{pmatrix} \tag{3}$$

Note that if $\Gamma_2 < 0$ *then* $d_1 P_z^1 A_z^1 + (*) + d_1 \dot{P}_z^1 < 0$, *and if* $\Gamma_{n_\mathrm{s}} < 0$ *then* $d_l P_l A_z^l + (*) + d_l \dot{P}_z^l < 0$. *Consequently,* $\dot{V} < 0$ *if* $\Gamma_l < 0$, *for* $l = 2, \ldots, n_\mathrm{s}$. *Hence, the string-connected system* (2) *is locally asymptotically stable, if* $\Gamma_l < 0$, *for* $l = 2, \ldots, n_\mathrm{s}$. □

Remark: The conditions of Theorem 1 implicitly rely on the assumption that the stability of the individual subsystems $\dot{\boldsymbol{x}}_l = A_z^l \boldsymbol{x}_l$, $l = 1, \ldots, n_\mathrm{s}$, is provable by the Lyapunov function $V_l = \boldsymbol{x}_l^T P_z^l \boldsymbol{x}_l$. If this does not hold, one can verify whether the subsystem in question taken together with one of its neighbors is stable and consider them as one subsystem.

Theorem 1 above explicitly states that the string-connected distributed TS fuzzy system is locally asymptotically stable, if each subsystem is stable and each pair of adjacent subsystems is stable. To develop conditions for each subsystem, i.e., circumvent the coupling between the neighboring subsystems, consider the condition $\Gamma_l < 0$, $l = 2, \ldots, n_\mathrm{s}$, with Γ_l defined as in (3). This condition can be written as

$$\Gamma_l = \begin{pmatrix} d_{l-1}P_z^{l-1}A_z^{l-1} + (*) + d_{l-1}\dot{P}_z^{l-1} + \delta I & 2d_{l-1}P_z^{l-1}A_z^{l-1,l} \\ (*) & -\delta I \end{pmatrix}$$
$$+ \begin{pmatrix} -\delta I & (*) \\ 2d_l P_z^l A_z^{l,l-1} & d_l P_z^l A_z^l + (*) + d_l \dot{P}_z^l + \delta I \end{pmatrix} < 0$$

for some $\delta > 0$. Moreover, we have

$$\Gamma_{l+1} = \begin{pmatrix} d_l P_z^l A_z^l + (*) + d_l \dot{P}_z^l + \delta I & 2d_l P^l A_z^{l,l+1} \\ (*) & -\delta I \end{pmatrix}$$
$$+ \begin{pmatrix} -\delta I & (*) \\ 2d_{l+1}P_z^{l+1}A_z^{l+1,l} & d_{l+1}P_z^{l+1}A_z^{l+1} + (*) + d_{l+1}\dot{P}_z^{l+1} + \delta I \end{pmatrix} < 0$$

By imposing that both terms concerning P_z^l in the above expressions are negative definite and by introducing $\delta_l = \delta / d_l$, $l = 1, 2, \ldots, n_\mathrm{s}$, the following conditions can be formulated:

Theorem 2. *The string-connected TS system* (2) *is locally asymptotically stable if there exist matrices* $P_i^l = (P_i^l)^T > 0$, $i = 1, 2, \ldots, m_l$, *and scalars* $\delta_l > 0$, $l = 1, \ldots, n_\mathrm{s}$, *so that*

$$\begin{aligned} &\begin{pmatrix} P_z^l A_z^l + (*) + \dot{P}_z^l + \delta_l I & 2P_z^l A_z^{l,l+1} \\ (*) & -\delta_l I \end{pmatrix} < 0 \\ &\begin{pmatrix} P_z^l A_z^l + (*) + \dot{P}_z^l + \delta_l I & 2P_z^l A_z^{l,l-1} \\ (*) & -\delta_l I \end{pmatrix} < 0 \\ &\textit{for } l = 1, \ldots, n_\mathrm{s} \end{aligned} \tag{4}$$

In the matrices above, with a redefinition of $\delta_l = \delta/d_l$, d_l is omitted, as it is positive and it appears in all terms.

It has to be noted that although Theorem 2 is in principle more conservative than Theorem 1, the conditions of Theorem 2 are decoupled for the subsystems.

The problem of stability analysis is now reduced to verifying the conditions of Theorem 2. Since the conditions depend on $\dot{P}_z^l$, in general they can be hard to verify. However, depending on how exactly the TS model has been obtained, the properties of the model and the purpose of the analysis, several results available in the literature can be employed. Some of the approaches will be discussed below.

3.2. Examples and discussion

In what follows, we present the conditions for different choices of the Lyapunov function. The simplest case is naturally when $P_i^l = P^l$, $i = 1, 2, \ldots, m_l$, i.e., a composite quadratic Lyapunov function is used. Since $P_z^l = \sum_{i=1}^{m_l} w_i(\boldsymbol{z})P_i^l = \sum_{i=1}^{m_l} w_i(\boldsymbol{z})P^l = P^l$, the conditions are reduced (4) to

$$\begin{aligned}
&\begin{pmatrix} P^l A_z^l + (*) + \delta_l I & 2P^l A_z^{l,l+1} \\ (*) & -\delta_l I \end{pmatrix} < 0 \\
&\begin{pmatrix} P^l A_z^l + (*) + \delta_l I & 2P^l A_z^{l,l-1} \\ (*) & -\delta_l I \end{pmatrix} < 0 \\
&\text{for } l = 1, \ldots, n_\text{s}
\end{aligned}$$

and actually to

$$\begin{aligned}
&\begin{pmatrix} P^l A_i^l + (*) + \delta_l I & 2P^l A_i^{l,l+1} \\ (*) & -\delta_l I \end{pmatrix} < 0 \\
&\begin{pmatrix} P^l A_i^l + (*) + \delta_l I & 2P_l A_i^{l,l-1} \\ (*) & -\delta_l I \end{pmatrix} < 0 \\
&\text{for } i = 1, 2, \ldots, m_l \quad l = 1, \ldots, n_\text{s}
\end{aligned}$$

Local asymptotic stability of the distributed TS system can be verified using Theorem 1 in [8], and applying relaxations such as [16, 2, 28]. A practical, although conservative possibility is using the relaxation in [28], the conditions becoming

find $P_i^l = (P_i^l)^T > 0$, $i = 1, 2, \ldots, m_l$, *and scalars* $\delta_l > 0$, $l = 1, \ldots, n_\text{s}$, *so that*

$$\begin{aligned}
&\Gamma_{ii}^{l+} < 0 \\
&\frac{2}{m_l - 1}\Gamma_{ii}^{l+} + \Gamma_{ij}^{l+} + \Gamma_{ji}^{l+} < 0 \\
&\Gamma_{ii}^{l-} < 0 \\
&\frac{2}{m_l - 1}\Gamma_{ii}^{l-} + \Gamma_{ij}^{l-} + \Gamma_{ji}^{l-} < 0
\end{aligned} \tag{5}$$

for $i, j = 1, 2, \ldots, m_l$, $l = 1, \ldots, n_{\mathrm{s}}$, *where*

$$\Gamma_{ij}^{l+} = \begin{pmatrix} P_i^l A_j^l + (*) + \delta_l I & 2P_i^l A_j^{l,l+1} \\ (*) & -\delta_l I \end{pmatrix}$$
$$\Gamma_{ij}^{l-} = \begin{pmatrix} P_i^l A_j^l + (*) + \delta_l I & 2P_i^l A_j^{l,l-1} \\ (*) & -\delta_l I \end{pmatrix}$$

It is important to note that thanks to conditions existing in the literature [29, 30, 31, 32] for local asymptotic stability, asymptotically necessary and sufficient conditions for (4) to hold can be derived as follows. It can be easily seen that with Theorem 1 of [8], the conditions (4) for local asymptotic stability result in the classical form

$$\sum_{j_1=1}^{r} \sum_{j_2=1}^{r} w_{j_1}(\boldsymbol{z}) w_{j_2}(\boldsymbol{z}) \Gamma_{j_1 j_2} < 0 \tag{6}$$

where r denotes a generic number of rules.

Using the matrix Pólya theorem (see [29, 30]), the necessary and sufficient condition for (6) obtained from (4) to hold for any fuzzy partition w is that there exists a large enough n so that

$$\tilde{\Gamma}_{\boldsymbol{i}} = \sum_{\boldsymbol{j} \in \mathcal{P}(\boldsymbol{i})} \Gamma_{j_1 j_2} = \sum_{\boldsymbol{j}=[j_1, j_2, \ldots, j_n]} w_{j_1}(\boldsymbol{z}) w_{j_2}(\boldsymbol{z}) \ldots w_{j_n}(\boldsymbol{z}) \Gamma_{j_1 j_2} < 0 \qquad \forall \boldsymbol{i} \in \mathcal{I}_n^+ \tag{7}$$

where $\tilde{\Gamma}_{\boldsymbol{i}}$ is the extension of (6) to n sums, $\mathcal{I}_n^+$ denotes the set of n-tuples (multiindices) of $\boldsymbol{i}$, and $\mathcal{P}(\boldsymbol{i})$ denotes the set of permutations of the multiindex $\boldsymbol{i}$. To reduce the number of decision variables, one can use e.g., the result in [32].

Compared to the results obtained in the discrete-time case using nonquadratic Lyapunov functions, in the continuous-time case, the number of existing results is extremely small. However, due to the conditions developed above, the latter results are all directly applicable for the conditions of Theorem 2. Depending on the knowledge of the system and how the TS model has been obtained, different results can be applied. For instance, given a bound on the derivatives of the membership functions, the result in [7] or if the TS model has been obtained by using the sector nonlinearity approach, the results in [8] can be directly employed.

Thanks to the conditions being developed using a nonquadratic Lyapunov function, we are now able to prove local asymptotic stability of a distributed system. Such a case is illustrated in the following example.

Example 1. *Consider the string-connected TS model composed of* $n_{\mathrm{s}} = 3$ *subsystems, each* *subsystem having two rules, i.e.,* $m_l = 2$, $l = 1, 2, 3$, *with the local matrices being*

$$
\begin{aligned}
A_1^1 &= \begin{pmatrix} -15.10 & -0.66 \\ 0.10 & -15.34 \end{pmatrix} & A_2^1 &= \begin{pmatrix} -6.78 & -2.08 \\ 4.75 & -8.58 \end{pmatrix} \\
A_1^{1,2} &= \begin{pmatrix} 0.02 & 0.13 \\ -0.03 & -0.08 \end{pmatrix} & A_2^{1,2} &= \begin{pmatrix} -0.06 & -0.10 \\ -0.05 & 0.02 \end{pmatrix} \\
A_1^2 &= \begin{pmatrix} -7.40 & 1.74 \\ -3.78 & -5.07 \end{pmatrix} & A_2^2 &= \begin{pmatrix} -17.02 & 3.26 \\ 3.05 & -10.92 \end{pmatrix} \\
A_1^{2,1} &= \begin{pmatrix} 0.01 & -0.08 \\ -0.12 & -0.04 \end{pmatrix} & A_2^{2,1} &= \begin{pmatrix} 0.02 & 0.04 \\ 0.12 & -0.11 \end{pmatrix} \\
A_1^{2,3} &= \begin{pmatrix} 0.13 & -0.01 \\ 0.03 & -0.01 \end{pmatrix} & A_2^{2,3} &= \begin{pmatrix} -0.07 & -0.28 \\ -0.05 & -0.01 \end{pmatrix} \\
A_1^3 &= \begin{pmatrix} -0.16 & 0.18 \\ -4.93 & -0.83 \end{pmatrix} & A_2^3 &= \begin{pmatrix} -2.45 & -1.20 \\ -4.03 & -7.83 \end{pmatrix} \\
A_1^{3,2} &= \begin{pmatrix} 0.14 & 0.08 \\ -0.13 & 0.07 \end{pmatrix} & A_2^{3,2} &= \begin{pmatrix} -0.04 & -0.002 \\ -0.04 & 0.13 \end{pmatrix}
\end{aligned}
$$

The quadratic stability of this system cannot be proven, even if considering it a centralized system. However, local asymptotic stability for the string-connected system can be proven using the conditions of Theorem 2 and applying the relaxation of [28], i.e., solving (5).

4. Observer design

Let us now consider observer design for string-connected TS models. We assume that the measurements do not depend on the states of the other subsystems, and that the scheduling variables $\boldsymbol{z}_l$ are available and they can be readily used in the observer. Also, the estimates of the neighboring systems are communicated, i.e., $\widehat{\boldsymbol{x}}_{l-1}$ and $\widehat{\boldsymbol{x}}_{l+1}$ can be used in the observer of subsystem l, $l = 1, 2, \ldots, n_{\mathrm{s}}$. Note that without this assumption, the states of the neighboring subsystems have to be considered unknown inputs and have to be estimated too.

The observer we consider is

$$
\begin{aligned}
\dot{\widehat{\boldsymbol{x}}}_l &= A_z^l \widehat{\boldsymbol{x}}_l + A_z^{l,l-1} \widehat{\boldsymbol{x}}_{l-1} + A_z^{l,l+1} \widehat{\boldsymbol{x}}_{l+1} + P_z^{-l} L_z^l (\boldsymbol{y}_l - \widehat{\boldsymbol{y}}_l) \\
\widehat{\boldsymbol{y}}_l &= C_z^l \widehat{\boldsymbol{x}}_l
\end{aligned}
\tag{8}
$$

for the lth subsystem, where $P_i^l = (P_i^l)^T > 0$, and L_i^l, $i = 1, 2, \ldots, m_l$, $l = 1, 2, \ldots, n_{\mathrm{s}}$ are matrices to be determined.

Consequently, the error dynamics of the lth subsystem are

$$
\dot{\boldsymbol{e}}_l = (A_z^l - P_z^{-l} L_z^l C_z^l) \boldsymbol{e}_l + A_z^{l,l-1} \boldsymbol{e}_{l-1} + A_z^{l,l+1} \boldsymbol{e}_{l+1} \tag{9}
$$

for $l = 1, 2, \ldots, n_{\mathrm{s}}$, which is again a string-connected system.

4.1. Design conditions

For the error dynamics (9) obtained by using the observer (8), the following result can be stated:

Theorem 3. *The error dynamics* (9) *are locally asymptotically stable, if there exist* $P_i^l = (P_i^l)^T > 0$, L_i^l, $i = 1, 2, \ldots, m_l$, $\delta_l > 0$, $l = 1, 2, \ldots, n_\mathrm{s}$ *such that*

$$\begin{aligned} &\begin{pmatrix} P_z^l A_z^l - L_z^l C_z^l + (*) + \dot{P}_z^l + \delta_l I & 2P_z^l A_z^{l,l+1} \\ (*) & -\delta_l I \end{pmatrix} < 0 \\ &\begin{pmatrix} P_z^l A_z^l - L_z^l C_z^l + (*) + \dot{P}_z^l + \delta_l I & 2P_z^l A_z^{l,l-1} \\ (*) & -\delta_l I \end{pmatrix} < 0 \\ &\textit{for } l = 1, \ldots, n_\mathrm{s} \end{aligned} \tag{10}$$

Proof. *Let us first consider the distributed system as a whole. Then, the dynamics can be written as*[2]

$$\begin{aligned} \dot{\boldsymbol{x}} &= \mathbb{A}_z \boldsymbol{x} + \mathbb{B}_z \boldsymbol{u} \\ \boldsymbol{y} &= \mathbb{C}_z \boldsymbol{x} \end{aligned} \tag{11}$$

where

$$\boldsymbol{x} = [\boldsymbol{x}_1^T, \boldsymbol{x}_2^T, \ldots, \boldsymbol{x}_{n_\mathrm{s}}^T]^T \quad \boldsymbol{u} = [\boldsymbol{u}_1^T, \boldsymbol{u}_2^T, \ldots, \boldsymbol{u}_{n_\mathrm{s}}^T]^T \quad \boldsymbol{y} = [\boldsymbol{y}_1^T, \boldsymbol{y}_2^T, \ldots, \boldsymbol{y}_{n_\mathrm{s}}^T]^T$$

$$\mathbb{A}_z = \begin{pmatrix} A_z^1 & A_z^{12} & 0 & \cdots & 0 & 0 \\ A_z^{21} & A_z^2 & A_z^{23} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A_z^{n_\mathrm{s},n_\mathrm{s}-1} & A_z^{n_\mathrm{s},n_\mathrm{s}} \end{pmatrix}$$

$$\mathbb{B}_z = \operatorname{diag}\left(B_z^1, B_z^2, \cdots, B_z^{n_\mathrm{s}}\right) \quad \mathbb{C}_z = \operatorname{diag}\left(C_z^1, C_z^2, \cdots, C_z^{n_\mathrm{s}}\right)$$

Consider for the centralized system (11) *the observer*

$$\begin{aligned} \dot{\widehat{\boldsymbol{x}}} &= \mathbb{A}_z \widehat{\boldsymbol{x}} + \mathbb{B}_z \boldsymbol{u} + \mathbb{P}_z^{-1} \mathbb{L}_z (\boldsymbol{y} - \widehat{\boldsymbol{y}}) \\ \widehat{\boldsymbol{y}} &= \mathbb{C}_z \widehat{\boldsymbol{x}} \end{aligned} \tag{12}$$

with $\mathbb{P}_z$ *and* $\mathbb{L}_z$ *being block-diagonal matrices of the form*

$$\begin{aligned} \mathbb{P}_z &= \operatorname{diag}\left(d_1 P_z^1, d_2 P_z^2, \cdots, d_{n_\mathrm{s}} P_z^{n_\mathrm{s}}\right) \\ \mathbb{L}_z &= \operatorname{diag}\left(d_1 L_z^1, d_2 L_z^2, \cdots, d_{n_\mathrm{s}} L_z^{n_\mathrm{s}}\right) \end{aligned}$$

with $d_i > 0$ *and* $P_i^l = (P_i^l)^T > 0$, $i = 1, 2, \ldots, m_l$, $l = 1, 2, \ldots, n_\mathrm{s}$. *It has to be noted that with this choice of the matrices, the structure of the estimation error remains block-tridiagonal, and therefore we can make use of the results obtained for stability analysis. Moreover, although*

[2]diag denotes a block-diagonal matrix composed of the arguments.

both $\mathbb{P}_z$ and $\mathbb{L}_z$ contain the scalars d_l, $l = 1, 2, \ldots, n_\mathrm{s}$, which offer an extra degree of freedom when solving the conditions, these scalars are not required in the observer.

Assuming that the scheduling variables do not depend on states that have to be estimated, we have the following estimation error dynamics for the whole system

$$\dot{\boldsymbol{e}} = \dot{\boldsymbol{x}} - \dot{\hat{\boldsymbol{x}}} = (\mathbb{A}_z - \mathbb{P}_z^{-1}\mathbb{L}_z\mathbb{C}_z)\boldsymbol{e} \tag{13}$$

with $\mathbb{A}_z - \mathbb{P}_z^{-1}\mathbb{L}_z\mathbb{C}_z$ having the block-tridiagonal form

$$\mathbb{A}_z - \mathbb{P}_z^{-1}\mathbb{L}_z\mathbb{C}_z = \begin{pmatrix} A_z^1 - (P_z^1)^{-1}L_z^1C_z^1 & A_z^{12} & 0 & \cdots & 0 & 0 \\ A_z^{21} & A_z^2 - (P_z^2)^{-1}L_z^2C_z^2 & A_z^{23} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A_z^{n_\mathrm{s},n_\mathrm{s}-1} & A_z^{n_\mathrm{s},n_\mathrm{s}} - (P_z^{n_\mathrm{s}})^{-1}L_z^{n_\mathrm{s}}C_z^{n_\mathrm{s}} \end{pmatrix} \tag{14}$$

Consider now the Lyapunov function $V = 2\boldsymbol{e}^T\mathbb{P}_z\boldsymbol{e}$. Its derivative is

$$\begin{aligned} \dot{V} &= 2\boldsymbol{e}^T(\mathbb{P}_z(\mathbb{A}_z - \mathbb{P}_z^{-1}\mathbb{L}_z\mathbb{C}_z) + (*) + \dot{\mathbb{P}}_z)\boldsymbol{e} \\ &= 2\boldsymbol{e}^T(\mathbb{P}_z\mathbb{A}_z - \mathbb{L}_z\mathbb{C}_z + (*) + \dot{\mathbb{P}}_z)\boldsymbol{e} \\ &= 2\boldsymbol{e}^T \begin{pmatrix} G_{z,z}^1 & (*) & 0 & \cdots & 0 & 0 \\ G_{z,z}^{2,1} + (G_{z,z}^{1,2})^T & G_{z,z}^2 & (*) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & G_{z,z}^{n_\mathrm{s},n_\mathrm{s}-1} + (G_{z,z}^{n_\mathrm{s},n_\mathrm{s}+1})^T & G_{z,z}^{n_\mathrm{s}} \end{pmatrix} \boldsymbol{e} \end{aligned}$$

where $G_{z,z}^l = d_lP_z^lA_z^l - d_lL_z^lC_z^l + (*) + d_l\dot{P}_z^l$, $G_{z,z}^{l,l-1} = d_lP_z^lA_z^{l,l-1}$, $G_{z,z}^{l,l+1} = (d_lP_z^lA_z^{l,l+1})^T$, $l = 1, 2, \ldots, n_\mathrm{s}$. Following the same reasoning as in Theorems 1 and 2, i.e., adding and subtracting $-2\delta I$ we obtain

$$\dot{V} = \sum_{l=1}^{n_\mathrm{s}} \begin{pmatrix} \boldsymbol{e}_{l-1} \\ \boldsymbol{e}_l \\ \boldsymbol{e}_{l+1} \end{pmatrix}^T \begin{pmatrix} -\delta I & (*) & 0 \\ 2G_{z,z}^{l,l-1} & 2G_{z,z}^l + 2\delta I & (*) \\ 0 & 2(G_{z,z}^{l,l+1})^T & -\delta I \end{pmatrix} \begin{pmatrix} \boldsymbol{e}_{l-1} \\ \boldsymbol{e}_l \\ \boldsymbol{e}_{l+1} \end{pmatrix}$$

which, further decoupled, leads to the conditions (10).

4.2. Examples and discussion

While for nonquadratic stability of TS systems some results exist in the literature, considerably fewer results are available for observer design. In what follows, we discuss what results can be applied for the proposed conditions.

Similarly to stability analysis, the simplest case is when the Lyapunov function is a composite quadratic one, and P_z is reduced to P. On the resulting conditions, several relaxations can be used, including the results of [16, 2, 33, 34, 28, 5]. For instance, using the results of [28], we have

Corollary 1. *The estimation error dynamics of the observer* (8) *are globally asymptotically stable if there exist matrices* $P_z^l = P_z^{lT} > 0$, L_i^l, *and scalars* $\delta_l > 0$, $i = 1, 2, \dots, m_l$, $l = 1, 2, \dots, n_{\mathrm{s}}$, *so that* (5) *holds with*

$$\Gamma_{ij}^{l+} = \begin{pmatrix} P^l A_i^l - L_i^l C_j^l + (*) + \delta_l I & 2P^l A_i^{l,l+1} \\ (*) & -\delta_l I \end{pmatrix}$$
$$\Gamma_{ij}^{l-} = \begin{pmatrix} P^l A_i^l - L_i^l C_j^l + (*) + \delta_l I & 2P^l A_i^{l,l-1} \\ (*) & -\delta_l I \end{pmatrix}$$

Existing results are again directly applicable for Theorem 3. Local asymptotic stability of the error dynamics can be verified using Theorem 1 of [8], and applying relaxations such as [16, 2, 28]. Given a bound on the derivatives of the membership functions, the result of [3, 7] can be properly modified for this case. If the TS model has been obtained by using the sector nonlinearity approach, the results of [35] can be directly applied.

An important result is that, similarly to stability analysis, asymptotically necessary and sufficient conditions can be stated for the conditions of Theorem 3 for the local asymptotic stability of the estimation error dynamics. Using the result of [8], the conditions of Theorem 3 are again of the classical form (6). By the extension of the matrices to n sums, the results from [30, 29] can be applied.

Example 2. *Consider the string-connected TS model composed of* $n_{\mathrm{s}} = 3$ *subsystems, each subsystem having* $m_l = 2$, $l = 1, 2, 3$ *rules, with the local matrices being*

$$A_1^1 = \begin{pmatrix} -23.68 & -1.92 \\ -6.25 & -2.50 \end{pmatrix} \qquad A_2^1 = \begin{pmatrix} -10.27 & 2.83 \\ 1.40 & 3.44 \end{pmatrix}$$
$$A_1^{1,2} = \begin{pmatrix} -0.06 & 0.01 \\ 0.03 & 0.13 \end{pmatrix} \qquad A_2^{1,2} = \begin{pmatrix} 0.15 & 0.14 \\ -0.05 & 0.02 \end{pmatrix}$$
$$A_1^2 = \begin{pmatrix} -5.9 & 2.37 \\ -3.7 & -6.4 \end{pmatrix} \qquad A_2^2 = \begin{pmatrix} -9.32 & 3.66 \\ -9.40 & -1.61 \end{pmatrix}$$
$$A_1^{2,1} = \begin{pmatrix} 0.18 & 0.15 \\ 0.20 & 0.06 \end{pmatrix} \qquad A_2^{2,1} = \begin{pmatrix} -0.02 & 0.20 \\ 0.04 & -0.05 \end{pmatrix}$$
$$A_1^{2,3} = \begin{pmatrix} -0.15 & -0.16 \\ -0.15 & 0.01 \end{pmatrix} \qquad A_2^{2,3} = \begin{pmatrix} -0.27 & -0.16 \\ 0.07 & 0.03 \end{pmatrix}$$
$$A_1^3 = \begin{pmatrix} -1.33 & -2.33 \\ 9.66 & -7.57 \end{pmatrix} \qquad A_2^3 = \begin{pmatrix} -5.73 & -6.5 \\ -3.53 & -11.39 \end{pmatrix}$$
$$A_1^{3,2} = \begin{pmatrix} 0.06 & -0.1 \\ -0.06 & 0.03 \end{pmatrix} \qquad A_2^{3,2} = \begin{pmatrix} -0.03 & -0.08 \\ 0.001 & 0.001 \end{pmatrix}$$

For each subsystem, the first state is measured, i.e., $C^l = \begin{pmatrix} 1 & 0 \end{pmatrix}$, $l = 1, 2, 3$.

Considering the system above as a centralized one, a quadratic Lyapunov function can be used to design an observer for it. However, for the string-connected system, the conditions *of Theorem 3 when considering a composite quadratic Lyapunov function are infeasible. A locally asymptotically stable observer can be designed using the conditions of Theorem 3 by solving the conditions* (5) *with*

$$\Gamma_{ij}^{l+} = \begin{pmatrix} P_i^l A_j^l - L_i^l C_j^l + (*) + \delta_l I & 2P_i^l A_j^{l,l+1} \\ (*) & -\delta_l I \end{pmatrix}$$
$$\Gamma_{ij}^{l-} = \begin{pmatrix} P_i^l A_j^l - L_i^l C_j^l + (*) + \delta_l I & 2P_i^l A_j^{l,l-1} \\ (*) & -\delta_l I \end{pmatrix}$$

for $i, j = 1, 2,\ l = 1, 2, 3.$

Solving[3] *the conditions above we obtain*

$$\begin{array}{llll}
P_1^1 = \begin{pmatrix} 1.66 & -1.02 \\ -1.02 & 0.84 \end{pmatrix} & P_2^1 = \begin{pmatrix} 1.10 & -0.54 \\ -0.54 & 0.35 \end{pmatrix} & L_1^1 = \begin{pmatrix} -32.10 \\ 17.59 \end{pmatrix} & L_2^1 = \begin{pmatrix} -10.56 \\ 6.25 \end{pmatrix} \\
P_1^2 = \begin{pmatrix} 0.56 & 0.12 \\ 0.12 & 0.13 \end{pmatrix} & P_2^2 = \begin{pmatrix} 0.82 & -0.04 \\ -0.04 & 0.21 \end{pmatrix} & L_1^2 = \begin{pmatrix} -3.19 \\ -0.60 \end{pmatrix} & L_2^2 = \begin{pmatrix} -6.71 \\ 1.50 \end{pmatrix} \\
P_1^3 = \begin{pmatrix} 1.75 & -0.28 \\ -0.28 & 0.16 \end{pmatrix} & P_2^3 = \begin{pmatrix} 1.64 & -0.25 \\ -0.25 & 0.19 \end{pmatrix} & L_1^3 = \begin{pmatrix} -4.58 \\ 0.09 \end{pmatrix} & L_2^3 = \begin{pmatrix} -7.99 \\ -6.97 \end{pmatrix} \\
\delta_1 = 0.066 & \delta_2 = 0.56 & \delta_3 = 0.54 &
\end{array}$$

i.e., a locally asymptotically stable observer. Moreover, the region of stability of the estimation error can be determined taking into account the membership functions and bounds on their derivatives.

5. Controller design

We now consider the design of a stabilizing state-feedback controller for a string-connected TS system, with the l-th subsystem of the form:

$$\dot{\boldsymbol{x}}_l = A_z^l \boldsymbol{x}_l + A_z^{l,l-1} \boldsymbol{x}_{l-1} + A_z^{l,l+1} \boldsymbol{x}_{l+1} + B_z^l \boldsymbol{u}_l \tag{15}$$

for $l = 1, 2, \ldots, n_s$. We assume for this case that the scheduling variables do not depend on the input, and use a control law of the form

$$\boldsymbol{u}_l = -F_z^l P_z^{-l} \boldsymbol{x}_l \tag{16}$$

The closed-loop system dynamics are

$$\dot{\boldsymbol{x}}_l = \left(A_z^l - B_z^l F_z^l P_z^{-l}\right) \boldsymbol{x}_l + A_z^{l,l-1} \boldsymbol{x}_{l-1} + A_z^{l,l+1} \boldsymbol{x}_{l+1} \tag{17}$$

[3] To solve the LMI conditions, the SeDuMi solver within the Yalmip [36] toolbox has been used. The results given are rounded to two decimal places.

5.1. Design conditions

For the closed-loop system (17) the following result can be stated:

Theorem 4. *The closed-loop system* (17) *is locally asymptotically stable if there exist matrices* $P_i^l = P_i^{lT} > 0$, F_i^l, $i = 1, 2, \ldots, m_l$, *and scalars* $\delta_l > 0$, $l = 1, 2, \ldots, n_s$, *such that*

$$\begin{aligned} &\begin{pmatrix} A_z^l P_z^l - B_z^l F_z^l + (*) - \dot{P}_z^l + \delta_l I & 2(A_z^{l,l+1} P_z^l)^T \\ (*) & -\delta_l I \end{pmatrix} < 0 \\ &\begin{pmatrix} A_z^l P_z^l - B_z^l F_z^l + (*) - \dot{P}_z^l + \delta_l I & 2(A_z^{l,l-1} P_z^l)^T \\ (*) & -\delta_l I \end{pmatrix} < 0 \end{aligned} \tag{18}$$

for $l = 1, 2, \ldots, n_s$.

Proof. *The full system can be written in a centralized form as*

$$\dot{\boldsymbol{x}} = \mathbb{A}_z \boldsymbol{x} + \mathbb{B}_z \boldsymbol{u}$$

where $\mathbb{A}_z$ *is block-tridiagonal and* $\mathbb{B}_z$ *is block-diagonal, similarly to* (11)*. The control law for the centralized system is*

$$\boldsymbol{u} = -\mathbb{F}_z \mathbb{P}_z^{-1} \boldsymbol{x}$$

with $\mathbb{F}_z$ *and* $\mathbb{P}_z$ *block-diagonal*

$$\begin{aligned} \mathbb{F}_z &= \operatorname{diag}\left(d_1 F_z^1,\, d_2 F_z^2,\, \cdots,\, d_{n_s} F_z^{n_s}\right) \\ \mathbb{P}_z &= \operatorname{diag}\left(d_1 P_z^1,\, d_2 P_z^2,\, \cdots,\, d_{n_s} P_z^{n_s}\right) \end{aligned}$$

$d_l > 0$, *and* $P_i^l = P_i^{lT} > 0$, $i = 1, 2, \ldots, m_l$, $l = 1, 2, \ldots, n_s$. *The resulting closed-loop dynamics are given by*

$$\dot{\boldsymbol{x}} = (\mathbb{A}_z - \mathbb{B}_z \mathbb{F}_z \mathbb{P}_z^{-1}) \boldsymbol{x}$$

Consider the Lyapunov function $V(\boldsymbol{x}) = 2\boldsymbol{x}^T \mathbb{P}_z^{-1} \boldsymbol{x}$*. The derivative* $\dot{V}$ *is obtained as*

$$\dot{V}(\boldsymbol{x}) = 2\boldsymbol{x}^T \left(\mathbb{P}_z^{-1} \left(\mathbb{A}_z - \mathbb{B}_z \mathbb{F}_z \mathbb{P}_z^{-1} \right) + (*) + \dot{\mathbb{P}}_z^{-1} \right) \boldsymbol{x}$$

and taking into account that $\mathbb{P}_z \dot{\mathbb{P}}_z^{-1} \mathbb{P}_z = -\dot{\mathbb{P}}_z$*, it can be written as*

$$\dot{V}(\boldsymbol{x}) = 2\boldsymbol{x}^T \mathbb{P}_z^{-1} \left(\mathbb{A}_z \mathbb{P}_z - \mathbb{B}_z \mathbb{F}_z + (*) - \dot{\mathbb{P}}_z \right) \mathbb{P}_z^{-1} \boldsymbol{x}$$

Given that $\mathbb{A}_z \mathbb{P}_z$ *is block-tridiagonal,* $\mathbb{B}_z \mathbb{F}_z$ *is block-diagonal and* $\dot{\mathbb{P}}_z$ *is block-diagonal, we obtain a similar relation as in Theorem 3:*

$$\dot{V}(\boldsymbol{x}) = 2\boldsymbol{x}^T \mathbb{P}_z^{-1} \begin{pmatrix} G_{z,z}^1 & (*) & 0 & \cdots & 0 & 0 \\ G_{z,z}^{2,1} + (G_{z,z}^{1,2})^T & G_{z,z}^2 & (*) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & G_{z,z}^{n_s,n_s-1} + (G_{z,z}^{n_s-1,n_s})^T & G_{z,z}^{n_s} \end{pmatrix} \mathbb{P}_z^{-1} \boldsymbol{x}$$

where $G^l_{z,z} = d_l A^l_z P^l_z - d_l B^l_z F^l_z + (*) - d_l \dot{P}^l_z$, $G^{l-1,l}_{z,z} = d_l A^{l-1,l}_z P^l_z$, $G^{l+1,l}_{z,z} = d_l A^{l+1,l}_z P^l_z$, $l = 1, 2, \ldots, n_\mathrm{s}$.

In the same manner as in Theorems 1 and 3, by adding and subtracting $2\delta I$ *a sufficient condition can be obtained as*

$$\begin{pmatrix} -\delta I & (*) & 0 \\ 2(G^{l-1,l}_{z,z})^T & G^l_{z,z} + 2\delta I & (*) \\ 0 & 2G^{l+1,l}_{z,z} & -\delta I \end{pmatrix} < 0$$

which can be further decoupled for each subsystem leading to conditions (18).

5.2. Examples and discussion

Similarly to stability analysis and observer design, existing results from the literature on non-quadratic stabilization of TS models readily apply with the developed conditions. If a composite quadratic Lyapunov function is used, P^l_z is reduced to P^l, and conditions similar to observer design are obtained. Of interest is the local asymptotic stabilization of string-connected TS models. For instance, using the results from [37], and the relaxation of [28], the conditions can be formulated as

find $P^l_i = P^{lT}_i > 0$, F^l_i, $i = 1, 2, \ldots, m_l$, *and scalars* $\delta_l > 0$, $l = 1, \ldots, n_\mathrm{s}$, *so that* (5) *holds with*

$$\begin{aligned} \Gamma^{l+}_{ij} &= \begin{pmatrix} A^l_i P^l_j - B^l_i F^l_j + (*) + \delta_l I & 2(A^{l,l+1}_i P^l_j)^T \\ (*) & -\delta_l I \end{pmatrix} < 0 \\ \Gamma^{l-}_{ij} &= \begin{pmatrix} A^l_i P^l_j - B^l_i F^l_j + (*) + \delta_l I & 2(A^{l,l-1}_i P^l_j)^T \\ (*) & -\delta_l I \end{pmatrix} < 0 \end{aligned} \tag{19}$$

Also, in this case, for the local asymptotic stability of the closed-loop system, asymptotically necessary and sufficient conditions can be stated for the conditions of Theorem 4. By the extension of the matrices obtained by applying Theorem 1 of [8] to n sums, the results from [30, 29] apply, and the conditions become asymptotically necessary.

Furthermore, depending on the knowledge on the derivatives of the membership functions, the results of [7] can be appropriately modified or those in [37] can be directly used. In what follows, we give an example for the local nonquadratic stabilization of string-connected TS systems.

Example 3. *Consider the string-connected TS model composed of* $n_\mathrm{s} = 3$ *subsystems, each subsystem having two rules, with the local matrices, adopted from Example 1 of [37], given by*

$$A^l_1 = \begin{pmatrix} 3.6 & -1.6 \\ 6.2 & -4.3 \end{pmatrix} \quad A^l_2 = \begin{pmatrix} -5 & -1.6 \\ 6.2 & -4.3 \end{pmatrix} \quad B^l_1 = \begin{pmatrix} -0.45 \\ -3 \end{pmatrix}, \quad B^l_2 = \begin{pmatrix} -1 \\ -3 \end{pmatrix}$$
$$A^{12}_i = A^{21}_i = A^{23}_i = A^{32}_i = 10^{-2} \cdot \begin{pmatrix} 2 & 1 \\ -5 & 10 \end{pmatrix}$$

for $l = 1, 2, 3$, $i = 1, 2$. *The system above is a homogenous string-connected system, i.e., the subsystems and the interconnection terms, respectively are the same.*

Using a quadratic Lyapunov function for the stabilization of the system results in unfeasible conditions. However, the system is locally asymptotically stabilizable using a nonquadratic Lyapunov function. Thus, we obtain

$$P_1^1 = \begin{pmatrix} 2.43 & 9.70 \\ 9.70 & 38.89 \end{pmatrix} \qquad P_2^1 = \begin{pmatrix} 3.48 & 7.56 \\ 7.56 & 18.12 \end{pmatrix} \quad F_1^1 = \begin{pmatrix} 11.08 & 30.33 \end{pmatrix} \qquad F_2^1 = \begin{pmatrix} 23.46 & 8.39 \end{pmatrix}$$

$$P_1^2 = \begin{pmatrix} 2.17 & 8.77 \\ 8.77 & 35.65 \end{pmatrix} \qquad P_2^2 = \begin{pmatrix} 3.13 & 6.63 \\ 6.63 & 15.35 \end{pmatrix} \quad F_1^2 = \begin{pmatrix} 10.16 & 27.69 \end{pmatrix} \qquad F_2^2 = \begin{pmatrix} 20.52 & 6.24 \end{pmatrix}$$

$$P_1^3 = \begin{pmatrix} 2.43 & 9.70 \\ 9.70 & 38.89 \end{pmatrix} \qquad P_2^3 = \begin{pmatrix} 3.48 & 7.56 \\ 7.56 & 18.12 \end{pmatrix} \quad F_1^3 = \begin{pmatrix} 11.08 & 30.33 \end{pmatrix} \qquad F_2^3 = \begin{pmatrix} 23.46 & 8.39 \end{pmatrix}$$

$$\delta_1 = 1.77, \quad \delta_2 = 1.64, \quad \delta_3 = 1.77$$

6. Conclusions

In this paper, we have considered stability analysis and controller and observer design for TS fuzzy systems connected in a string. Sufficient stability and observer and controller design conditions have been derived, and illustrated on numerical examples.

In this paper, we have considered a special class of distributed systems. In our future research, we will extend the results presented in this paper to more general, sparsely interconnected systems. Other special cases that deserve investigation are distributed system where the interconnections are symmetrical, or homogeneous distributed systems.

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