On non-PDC local observers for TS fuzzy systems

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***Abstract*— In this paper we propose a method to design local observers for Takagi-Sugeno fuzzy models obtained from nonlinear systems by the sector nonlinearity approach. When a global observer cannot be designed, using our method it is still possible to design observers that are valid in a well-defined region of the state-space. The design is based on a nonquadratic Lyapunov function. Depending on whether or not the scheduling vector is a function of the states to be estimated, the conditions are formulated as an LMI or a BMI problem, respectively. The results are illustrated on simulation examples, for which classical observer design conditions are unfeasible.**

I. Introduction

A large class of nonlinear systems can be exactly represented by Takagi-Sugeno (TS) fuzzy models [13]. The TS fuzzy model consists of a rule base, with the consequent of each rule being a linear or affine model. A constructive approach to obtain a TS model given a dynamic nonlinear system is the sector nonlinearity approach [10].

In order to analyze the stability of a TS fuzzy model, the direct Lyapunov approach has been used. Stability conditions have been derived using quadratic Lyapunov functions [12], [14], [15], piecewise continuous Lyapunov functions [3], [6], and more recently, nonquadratic Lyapunov functions [5], [8], [9].

For discrete-time TS systems, nonquadratic Lyapunov functions have been successfully employed. However, their use for continuous-time fuzzy systems has been scarce, due to the fact that the analysis and synthesis using nonquadratic Lyapunov functions involve the derivatives of the membership functions, and without further assumptions on the membership functions, they lead to very conservative bounds. However, latest results [4] prove that such Lyapunov functions are extremely useful for establishing local stability of an equilibrium point, and for approximating the domain of attraction for fuzzy models obtained by the sector nonlinearity approach [10].

In this paper we use nonquadratic Lyapunov functions to design local observers for TS systems obtained by the sector nonlinearity approach. For a fuzzy model, well-established methods and algorithms exist to design fuzzy observers. Several types of observers have been developed for continuous-time TS fuzzy systems, among which: fuzzy Thau-Luenberger observers [14], [16], reduced-order observers [1], [2], and sliding-mode observers [11]. Most of the stability and design conditions rely on the feasibility of an associated system of linear matrix inequalities (LMIs), in general derived from the negative definiteness of a quadratic Lyapunov function. Therefore, all these observers are globally valid. However, in general, nonlinear systems are not globally observable or detectable, and therefore, the design of a global observer is not always possible. Moreover, when using a quadratic Lyapunov function, the conditions are conservative.

In this paper, instead of designing a global observer, we design local observers. At the same time we also determine the domain where the observer is valid, i.e., where the estimation error converges to zero. Note that for the simplicity of the notation and computations, in this paper we only consider TS systems with a common measurement matrix. We distinguish two cases: 1) if the scheduling vector depends only on the measured variables, the design conditions can be formulated as LMIs; and 2) if the scheduling vector depends also on the states that have to be estimated, the approach has to be modified, and BMI conditions are obtained. Since in this paper we only consider observer design, without state or output feedback control, the input vector is treated simply as a measured input (not necessarily control input). From this point of view, the membership functions may depend on the input as it is considered a measured variable.

The paper is organized as follows. Section II presents the TS models obtained by the sector nonlinearity approach used in this paper and the stability conditions our method relies on. In Section III, the observer design conditions are derived, for the two cases: when the scheduling vector does not depend on the states to be estimated and when it does. The resulting conditions are summarized as LMI and BMI problems, respectively. Section IV illustrates the design methods on examples, and finally, Section V concludes the paper.

II. Preliminaries

Consider a nonlinear system of the form

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{z})\boldsymbol{x} + \boldsymbol{g}(\boldsymbol{z})\boldsymbol{u} \\ \boldsymbol{y} &= C\boldsymbol{x} \end{aligned} \tag{1}$$

with $\boldsymbol{f}$ and $\boldsymbol{g}$ smooth nonlinear matrix functions, $\boldsymbol{z}$ is the vector of scheduling variables, $\boldsymbol{z} = T[\boldsymbol{x}^{\mathrm{T}}\,\boldsymbol{y}^{\mathrm{T}}\,\boldsymbol{u}^{\mathrm{T}}]^{\mathrm{T}}$, T a constant matrix[1], $\boldsymbol{x} \in \mathcal{R}^n$ the state vector, $\boldsymbol{u} \in \mathcal{R}^{n_u}$ the input vector, and $\boldsymbol{y} \in \mathcal{R}^{n_y}$ the measurement vector, all variables assumed to be bounded on a compact set $\mathcal{C}_{xyu}$.

Let $\mathrm{nl}_j(\cdot) \in [\underline{\mathrm{nl}}_j, \overline{\mathrm{nl}}_j]$, $j = 1, 2, \ldots, p$ be the set of bounded nonlinearities in $\boldsymbol{f}$ and $\boldsymbol{g}$. Using the sector

[1]For the ease of notation, in this paper we only consider the case when the scheduling vector is a linear combination of states and measured variables. If the scheduling vector is a nonlinear function of these variables, similar, although more complex conditions can be derived.

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nonlinearity approach [10], an exact TS fuzzy representation of (1) can be obtained by constructing first the weighting functions

$$w_0^j(\cdot) = \frac{\overline{\mathrm{nl}}_j - \mathrm{nl}_j(\cdot)}{\overline{\mathrm{nl}}_j - \underline{\mathrm{nl}}_j} \quad w_1^j(\cdot) = 1 - w_0^j(\cdot) \quad j = 1, 2, \dots, p$$

and defining the membership functions as

$$h_i(\boldsymbol{z}) = \prod_{j=1}^{p} w_{i_j}^j(\boldsymbol{z}_j) \tag{2}$$

with $i = 1, 2, \cdots, 2^p$, $i_j \in \{0, 1\}$. Note that these membership functions are normal, i.e., $h_i(\boldsymbol{z}) \geq 0$ and $\sum_{i=1}^{r} h_i(\boldsymbol{z}) = 1$, $r = 2^p$, where r is the number of rules.

Then, an exact representation of (1) is given as:

$$\begin{aligned} \dot{\boldsymbol{x}} &= \sum_{i=1}^{r} h_i(\boldsymbol{z})(A_i\boldsymbol{x} + B_i\boldsymbol{u}) \\ \boldsymbol{y} &= C\boldsymbol{x} \end{aligned} \tag{3}$$

with $\boldsymbol{z} = T[\boldsymbol{x}^{\mathrm{T}}\, \boldsymbol{y}^{\mathrm{T}}\, \boldsymbol{u}^{\mathrm{T}}]^{\mathrm{T}}$, r the number of local linear models, A_i, B_i, $i = 1, 2, \dots, r$ matrices of proper dimensions, and h_i, $i = 1, 2, \dots, r$ defined as in (2).

Controller and observer design for such systems often leads to establishing the negative definiteness of double summations of the form $\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\boldsymbol{z})h_j(\boldsymbol{z})\Upsilon_{ij}$, with Υ_{ij} symmetric matrices of appropriate dimensions. In this paper we use the following relaxation [17]:

Theorem 1: Let Υ_{ij} be matrices of proper dimensions. Then,

$$\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\boldsymbol{z})h_j(\boldsymbol{z})\Upsilon_{ij} < 0 \tag{4}$$

holds, if

$$\begin{aligned} &\Upsilon_{ii} < 0 \qquad i = 1, 2, \dots, r, \\ &\frac{2}{r-1}\Upsilon_{ii} + \Upsilon_{ij} + \Upsilon_{ji} < 0, \qquad i, j = 1, 2, \dots, r,\ i \neq j \end{aligned} \tag{5}$$

Observer design for systems of the form (3) has been largely investigated in the literature, in general based on stability conditions for the dynamics of the estimation error. Since most existing stability conditions ensure global stability, a global observer is typically designed. However, since (3) is essentially a nonlinear system, it is possible, that the system (3) is not observable for every combination of ($\boldsymbol{x}$, $\boldsymbol{u}$, $\boldsymbol{y}$). In this paper, we extend the novel local asymptotic stability conditions reported in [4] to observer design.

In the sequel we denote $M_z = \sum_{i=1}^{r} h_i(\boldsymbol{z})M_i$, where M_i, $i = 1, 2, \dots, r$ are matrices. I and 0 denote the identity and zero matrices of proper dimensions, and $\mathcal{H}(X)$ denotes the Hermitian of the matrix X, $\mathcal{H}(X) = X + X^{\mathrm{T}}$.

The conditions in [4] have been developed for the autonomous TS system

$$\dot{\boldsymbol{x}} = \sum_{i=1}^{r} h_i(\boldsymbol{z})A_i\boldsymbol{x} \tag{6}$$

with h_i defined as in (2) and are formulated as

Theorem 2: [4] If $\mathcal{H}(P_zA_z) < 0$, has a solution $P_i = P_i^{\mathrm{T}} > 0$, $i = 1, 2, \dots, r$, then there exists a domain $\mathcal{D}$ including the origin such that the TS model (6) *is locally asymptotically stable.*

III. Observer design

Consider the TS fuzzy system (3), with the membership functions given as (2). Our goal is to design an observer of the form

$$\begin{aligned} \dot{\widehat{\boldsymbol{x}}} &= \sum_{i=1}^{r} h_i(\widehat{\boldsymbol{z}})(A_i\widehat{\boldsymbol{x}} + B_i\boldsymbol{u} + K_i(\widehat{\boldsymbol{z}})(\boldsymbol{y} - \widehat{\boldsymbol{y}})) \\ \widehat{\boldsymbol{y}} &= C\widehat{\boldsymbol{x}} \end{aligned} \tag{7}$$

and determine a domain $\mathcal{D}_z$ such that for any trajectory of $\boldsymbol{x}$ in $\mathcal{D}_z$, the estimation error converges to zero. In (7), $\widehat{\boldsymbol{z}}$ denotes the estimated scheduling vector, in the case when the scheduling vector is not measured.

In what follows, two cases are distinguished, depending on whether or not the scheduling vector depends on the states to be estimated.

A. State-independent scheduling vector

The simplest case in the observer design problem is when the scheduling vector depends only on measured variables. Therefore, consider first the case when the scheduling vector $\boldsymbol{z}$ does not depend on the states to be estimated, i.e., $\boldsymbol{z} = T[\boldsymbol{y}^{\mathrm{T}}\, \boldsymbol{u}^{\mathrm{T}}]^{\mathrm{T}}$, with T a selection matrix of the form $T = [T_1\ T_2]$, T_1 of dimension $n_z \times n_y$ and T_2 of dimension $n_z \times n_u$.

The TS system is given by (3), and the observer is of the form

$$\begin{aligned} \dot{\widehat{\boldsymbol{x}}} &= \sum_{i=1}^{r} h_i(\boldsymbol{z})(A_i\widehat{\boldsymbol{x}} + B_i\boldsymbol{u} + K_i(\boldsymbol{z})(\boldsymbol{y} - \widehat{\boldsymbol{y}})) \\ \widehat{\boldsymbol{y}} &= C\widehat{\boldsymbol{x}} \end{aligned} \tag{8}$$

with $K_i(\boldsymbol{z})$ to be determined.

The error dynamics are

$$\dot{\boldsymbol{e}} = \sum_{i=1}^{r} h_i(\boldsymbol{z})(A_i\boldsymbol{e} - K_i(\boldsymbol{z})C\boldsymbol{e}) \tag{9}$$

By using a Lyapunov function of the form

$$V = \boldsymbol{e}^{\mathrm{T}}\sum_{i=1}^{r} h_i(\boldsymbol{z})P_i\boldsymbol{e} = \boldsymbol{e}^{\mathrm{T}}P_z\boldsymbol{e}$$

with $P_i = P_i^{\mathrm{T}} > 0$, $i = 1, 2, \dots, r$, we have

$$\dot{V} = \boldsymbol{e}^{\mathrm{T}}\sum_{i=1}^{r} h_i(\boldsymbol{z})\mathcal{H}(P_z(A_i - K_i(\boldsymbol{z})C))\boldsymbol{e} + \boldsymbol{e}^{\mathrm{T}}\dot{P}_z\boldsymbol{e}. \tag{10}$$

Consider the first part of the derivative,

$$\dot{V}_c = \boldsymbol{e}^{\mathrm{T}}\sum_{i=1}^{r} h_i(\boldsymbol{z})\mathcal{H}(P_z(A_i - K_i(\boldsymbol{z})C))\boldsymbol{e}$$

The sum $\dot{V}_c$ is positive for any $\boldsymbol{e}$, if $\mathcal{H}(P_z(A_i - K_i(\boldsymbol{z})C))$ is positive definite. By choosing $K_i(\boldsymbol{z}) = P_z^{-1}L_i$, with L_i, $i = 1, 2, \ldots, r$ constant matrices, we have

$$\begin{aligned}\dot{V}_c =&\boldsymbol{e}^{\mathrm{T}} \sum_{i=1}^{r} h_i(\boldsymbol{z})\mathcal{H}(P_z(A_i - K_i(\boldsymbol{z})C))\boldsymbol{e}\\ =&\boldsymbol{e}^{\mathrm{T}} \sum_{i=1}^{r} h_i(\boldsymbol{z})\mathcal{H}(P_zA_i - L_iC)\boldsymbol{e}\\ =&\boldsymbol{e}^{\mathrm{T}} \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\boldsymbol{z})h_j(\boldsymbol{z})\mathcal{H}(P_jA_i - L_iC)\boldsymbol{e}\end{aligned}$$

in which case $\dot{V}_c < 0$ can be formulated as an LMI problem.

Consider now the second part of (10), i.e., $\dot{P}_z$. Similarly to the result in [4], we have

$$\dot{P}_z = \sum_{i=1}^{r}\sum_{j=1}^{p} h_i(\boldsymbol{z})\frac{\partial w_0^j}{\partial \boldsymbol{z}_j}(P_{g_1(i,j)} - P_{g_2(i,j)})\dot{\boldsymbol{z}}_j \tag{11}$$

where $\boldsymbol{z}_j$ denotes the jth element of the vector $\boldsymbol{z}$, with the indices of the P matrices computed as (see [4])

$$\begin{aligned}g_1(i,j) =&\lfloor (i-1)/2^{p+1-j})\rfloor \times 2^{p+1-j}\\ &+1 + (i-1) \mod 2^{p-j}\\ g_2(i,j) =&g_1(i,j) + 2^{p-j}\end{aligned} \tag{12}$$

Since $\boldsymbol{z} = T[\boldsymbol{y}^{\mathrm{T}},\ \boldsymbol{u}^{\mathrm{T}}]^{\mathrm{T}}$, with $T = [T_1\ T_2]$, we have

$$\begin{aligned}\dot{\boldsymbol{z}}_j =&\sum_{i=1}^{n_y}(T_1C)_{ji}\dot{\boldsymbol{y}}_i + \sum_{i=1}^{n_u}(T_2)_{ji+n_y}\dot{\boldsymbol{u}}_i\\ =&\sum_{i=1}^{n}(T_1C)_{ji}(\sum_{k=1}^{r} h_k(\boldsymbol{z})(\sum_{l=1}^{n}(A_k)_{il}\boldsymbol{x}_l + \sum_{l=1}^{n_u}(B_k)_{il}\boldsymbol{u}_l)\\ &+\sum_{i=1}^{n_u}(T_2)_{ji+n_y}\dot{\boldsymbol{u}}_i)\end{aligned}$$

where M_{il} denotes the (i,l)th element of the matrix M, or,

$$\begin{aligned}\dot{\boldsymbol{z}}_j =&\sum_{k=1}^{r} h_k(\boldsymbol{z})(\sum_{l=1}^{n}(T_1CA_k)_{jl}\boldsymbol{x}_l\\ &+\sum_{l=1}^{n_u}(T_1CB_k)_{jl}\boldsymbol{u}_l + \sum_{l=1}^{n_u}(T_2)_{jl+n_y}\dot{\boldsymbol{u}}_l)\end{aligned} \tag{13}$$

and consequently

$$\begin{aligned}\dot{P}_z =&\sum_{i=1}^{r}\sum_{j=1}^{p} h_i(\boldsymbol{z})\sum_{k=1}^{r} h_k(\boldsymbol{z})\Big[\sum_{l=1}^{n}\frac{\partial w_0^j}{\partial \boldsymbol{z}_j}\boldsymbol{x}_l(T_1CA_k)_{jl}\\ &+\sum_{l=1}^{n_u}\frac{\partial w_0^j}{\partial \boldsymbol{z}_j}\boldsymbol{u}_l(T_1CB_k)_{jl}\\ &+\sum_{l=1}^{n_u}\frac{\partial w_0^j}{\partial \boldsymbol{z}_j}\dot{\boldsymbol{u}}_l(T_2)_{jl+n_y}\Big](P_{g_1(i,j)} - P_{g_2(i,j)})\end{aligned} \tag{14}$$

For the ease of notation, denote $\frac{\partial w_0^j}{\partial z_j}\boldsymbol{x}_l$, $l = 1, \ldots, n$, $\frac{\partial w_0^j}{\partial z_j}\boldsymbol{u}_l$, $l = 1, \ldots, n_u$ and $\frac{\partial w_0^j}{\partial z_j}\dot{\boldsymbol{u}}_l$, $l = 1, \ldots, n_u$ by q_{jl}, $l = 1, \ldots, n + 2n_u$ and $(T_1CA_k)_{jl}$, $l = 1, \ldots, n$, $(T_1CB_k)_{jl}$, $l = 1, \ldots, n_u$, and $(T_2)_{jl+n_y}$, $l = 1, \ldots, n_u$, by M_{jl}, $l = 1, \ldots, n + 2n_u$, respectively. Then,

$$\dot{P}_z = \sum_{i=1}^{r} h_i(\boldsymbol{z})\sum_{j=1}^{p}\sum_{k=1}^{r} h_k(\boldsymbol{z})\sum_{l=1}^{n+2n_u} q_{jl}M_{jl}(P_{g_1(i,j)} - P_{g_2(i,j)}) \tag{15}$$

Knowing the bounds $|q_{jl}| < \lambda_{jl}$, $l = 1, \ldots, n + 2n_u$, and using the property that $Y + q_{jl}X \leq 0$, if

$$\begin{aligned}Y + \lambda_{jl}X \leq 0\\ Y - \lambda_{jl}X \leq 0\end{aligned} \tag{16}$$

LMI constraints can be formulated, as follows:

Theorem 3: If there exists matrices $P_i = P_i^{\mathrm{T}} > 0$, and L_i, $i = 1, 2, \ldots, r$ such that

$$\begin{aligned}&\Upsilon_{ii}^m < 0 \qquad i = 1, 2, \ldots, r,\ m = 1, \ldots, 2^{p\times(n+2n_u)}\\ &\frac{2}{r-1}\Upsilon_{ii}^m + \Upsilon_{ij}^m + \Upsilon_{ji}^m < 0\\ &i, j = 1, 2, \ldots, r,\ i \neq j,\ m = 1, \ldots, 2^{p\times(n+2n_u)}\end{aligned} \tag{17}$$

with

$$\begin{aligned}\Upsilon_{ij}^m =&P_iA_j - L_jC + A_j^{\mathrm{T}}P_i - C^{\mathrm{T}}L_j^{\mathrm{T}}\\ &+\sum_{k=1}^{p}\sum_{l=1}^{n+2n_u}(-1)^{d_{kl}^m}\lambda_{kl}M_{kl}(P_{g_1(i,k)} - P_{g_2(i,k)})\end{aligned} \tag{18}$$

hold, where d_{kl}^m are obtained from the binary representation of $m-1 = d_{pn+2n_u}^m + d_{pn+2n_u-1}^m \times 2 + \cdots + d_{11}^m \times 2^{n+2n_u}$ and $g_1(i,k)$, $g_2(i,k)$ are defined as in (12)*, then the estimation error $\mathbf{e}$ of the observer* (8) *tends to zero exponentially for any trajectory of $\boldsymbol{x}$ satisfying* (3) *and $\bigcap_{k,l}\{(\boldsymbol{x}, \boldsymbol{u}, \dot{\boldsymbol{u}}) : |q_{kl}| \leq \lambda_{kl}\}$.*

Proof: Follows immediately from Theorem 2 of [4].

Remark: Note that the condition derived concerns $\boldsymbol{x}$, since the local observability of the system depends on the actual states, not on the estimated states. It is also possible to derive conditions on $\widehat{\boldsymbol{x}}$, although those attract a condition on $\boldsymbol{e}$.

If one uses instead of (16) the property that for any symmetrical positive definite matrix S, $Y + q_{jl}X \leq Y + \frac{1}{2}(\lambda_{jl}^2 S + XS^{-1}X)$, then the results can be formulated as:

Theorem 4: If there exists matrices $P_i = P_i^{\mathrm{T}} > 0$, and L_i, $i = 1, 2, \ldots, r$, S_{kl}, $k = 1, 2, \ldots, p$, $l = 1, 2, \ldots, n+2n_u$ such that

$$\begin{aligned}&\Upsilon_{ii} < 0 \qquad i = 1, 2, \ldots, r,\ m = 1, \ldots, 2^{p\times(n+2n_u)}\\ &\frac{2}{r-1}\Upsilon_{ii} + \Upsilon_{ij} + \Upsilon_{ji} < 0, \qquad i, j = 1, 2, \ldots, r,\ i \neq j,\end{aligned}$$

with

$$\Upsilon_{ij} = \begin{pmatrix} G_{ij} & (*) & \cdots & (*) & \cdots & (*)\\ M_{11}\bar{P}_{(i,k)} & -2S_{11} & \cdots & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots\\ M_{p1}\bar{P}_{(i,k)} & 0 & \cdots & -2S_{p1} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots\\ M_{pn+2n_u}\bar{P}_{(i,k)} & 0 & \cdots & 0 & \cdots & -2S_{pn+2n_u}\end{pmatrix}$$

hold, with

$$G_{ij} = P_i A_j - L_j C + A_j^{\mathrm{T}} P_i - C^{\mathrm{T}} L_j^{\mathrm{T}} + \frac{1}{2}\sum_{k=1}^{p}\sum_{l=1}^{n+2n_u} \lambda_{kl}^2 S_{ku}$$

and $\bar{P}_{(i,k)} = P_{g_1(i,k)} - P_{g_2(i,k)}$ *where* $(*)$ *denotes the symmetric term and* $g_1(i,k)$, $g_2(i,k)$ *are defined as in* (12), *then the estimation error* $\boldsymbol{e}$ *of the observer* (8) *tends to zero exponentially for any trajectory of* $\boldsymbol{x}$ *satisfying* (3) *in the outermost Lyapunov level contained in* $\bigcap_{k,l}\{(\boldsymbol{x},\boldsymbol{u},\dot{\boldsymbol{u}}) : |q_{kl}| \leq \lambda_{kl}\}$.

Proof: Follows immediately from Theorem 3 of [4].

Remark: In general, the observer will not be valid on the whole domain $\mathcal{C}_{xyu}$ where the TS system is defined. In order to find estimates of the region where the observer is valid, the bounds λ_{kl} may be treated as variables and maximized. However, if λ_{kl} are treated as variables, a BMI problem is obtained.

B. State-dependent scheduling vector

In many applications, however, the scheduling vector depends also on states that need to be estimated. For the ease of notation, consider only $\boldsymbol{z} = T\boldsymbol{x}$, i.e., the scheduling vector is a linear combination of the states, although $\boldsymbol{z} = T[\boldsymbol{x}^{\mathrm{T}}\,\boldsymbol{y}^{\mathrm{T}}\,\boldsymbol{u}^{\mathrm{T}}]^{\mathrm{T}}$ can also be used. In this case, we don't have access to $\boldsymbol{z}$, and instead $\widehat{\boldsymbol{z}}$, the estimated scheduling variables have to be used in the observer.

Then, the TS system is again (3), but the observer becomes

$$\begin{aligned}\dot{\widehat{\boldsymbol{x}}} &= \sum_{i=1}^{r} h_i(\widehat{\boldsymbol{z}})(A_i\widehat{\boldsymbol{x}} + B_i\boldsymbol{u} + K_i(\widehat{\boldsymbol{z}})(\boldsymbol{y} - \widehat{\boldsymbol{y}}))\\ \widehat{\boldsymbol{y}} &= C\widehat{\boldsymbol{x}}\end{aligned} \tag{19}$$

with $K_i(\widehat{\boldsymbol{z}})$ to be designed. Since the state variables have to be estimated, K_i is also a function of $\widehat{\boldsymbol{z}}$, and not of $\boldsymbol{z}$.

The error dynamics are given as

$$\begin{aligned}\dot{\boldsymbol{e}} &= \sum_{i=1}^{r} h_i(\widehat{\boldsymbol{z}})(A_i\boldsymbol{e} - K_i(\widehat{\boldsymbol{z}})C\boldsymbol{e})\\ &\quad + \sum_{i=1}^{r}(h_i(\boldsymbol{z}) - h_i(\widehat{\boldsymbol{z}}))(A_i\boldsymbol{x} + B_i\boldsymbol{u})\end{aligned} \tag{20}$$

Note however, that in this case it is not possible to use simultaneously a Lyapunov function of the form $V = \boldsymbol{e}^{\mathrm{T}}\sum_{i=1}^{r} h_i(\widehat{\boldsymbol{z}})P_i\boldsymbol{e} = \boldsymbol{e}^{\mathrm{T}}P_{\widehat{\boldsymbol{z}}}\boldsymbol{e}$, with $P_i = P_i^{\mathrm{T}} > 0$, $i = 1, 2, \ldots, r$, and $K_i(\widehat{\boldsymbol{z}})$ chosen as $K_i(\widehat{\boldsymbol{z}}) = P_{\widehat{\boldsymbol{z}}}^{-1}L_i$, as it will be shown in what follows. If these functions are used, one obtains

$$\begin{aligned}\dot{V} =& \boldsymbol{e}^{\mathrm{T}}\sum_{i=1}^{r} h_i(\widehat{\boldsymbol{z}})\mathcal{H}(P_{\widehat{\boldsymbol{z}}}(A_i - K_i(\widehat{\boldsymbol{z}})C))\boldsymbol{e}\\ &+ \boldsymbol{e}^{\mathrm{T}}\sum_{i=1}^{r}\mathcal{H}(P_{\widehat{\boldsymbol{z}}}(h_i(\boldsymbol{z}) - h_i(\widehat{\boldsymbol{z}}))(A_i\boldsymbol{x} + B_i\boldsymbol{u})) + \boldsymbol{e}^{\mathrm{T}}\dot{P}_{\widehat{\boldsymbol{z}}}\boldsymbol{e}.\end{aligned}$$

Consider first the part $\dot{V}_c = \boldsymbol{e}^{\mathrm{T}}\sum_{i=1}^{r} h_i(\boldsymbol{z})\mathcal{H}(P_{\widehat{\boldsymbol{z}}}(A_i - K_i(\widehat{\boldsymbol{z}})C))\boldsymbol{e} + 2\boldsymbol{e}^{\mathrm{T}}\sum_{i=1}^{r} P_{\widehat{\boldsymbol{z}}}(h_i(\boldsymbol{z}) - h_i(\widehat{\boldsymbol{z}}))(A_i\boldsymbol{x} + B_i\boldsymbol{u})$. With the assumption[2] that $\sum_{i=1}^{r}(h_i(\boldsymbol{z}) - h_i(\widehat{\boldsymbol{z}}))(A_i\boldsymbol{x} + B_i\boldsymbol{u}) = \Delta\boldsymbol{e}$, with $\|\Delta\| \leq \lambda$ and $K_i(\widehat{\boldsymbol{z}}) = P_{\widehat{\boldsymbol{z}}}^{-1}L_i$, we have

$$\begin{aligned}\dot{V}_c =& \boldsymbol{e}^{\mathrm{T}}\sum_{i=1}^{r} h_i(\widehat{\boldsymbol{z}})\mathcal{H}(P_{\widehat{\boldsymbol{z}}}(A_i - K_i(\boldsymbol{z})C)) + \boldsymbol{e}^{\mathrm{T}}\sum_{i=1}^{r}\mathcal{H}(P_{\widehat{\boldsymbol{z}}}\Delta)\boldsymbol{e}\\ \leq& \boldsymbol{e}^{\mathrm{T}}\sum_{i=1}^{r} h_i(\widehat{\boldsymbol{z}})\mathcal{H}(P_{\widehat{\boldsymbol{z}}}A_i - L_iC + P_{\widehat{\boldsymbol{z}}}\Delta))\boldsymbol{e}\\ \leq& \boldsymbol{e}^{\mathrm{T}}\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\widehat{\boldsymbol{z}})h_j(\widehat{\boldsymbol{z}})\mathcal{H}(P_jA_i - L_iC + P_j\Delta))\boldsymbol{e}\end{aligned}$$

in which case $\dot{V}_c < 0$ can be formulated as an LMI problem.

However, when considering $\dot{P}_{\widehat{\boldsymbol{z}}}$, one obtains

$$\dot{P}_{\widehat{\boldsymbol{z}}} = \sum_{i=1}^{r}\sum_{j=1}^{p} h_i(\widehat{\boldsymbol{z}})\frac{\partial w_0^j}{\partial\widehat{\boldsymbol{z}}_j}(P_{g_1(i,j)} - P_{g_2(i,j)})(T\dot{\widehat{\boldsymbol{x}}})_j$$

with $g_1(i,j)$ and $g_2(i,j)$ defined in (12).

Now, $\dot{\widehat{\boldsymbol{x}}}_j$ is expressed as

$$\dot{\widehat{\boldsymbol{x}}}_j = \sum_{k=1}^{r} h_k(\widehat{\boldsymbol{z}})(\sum_{l=1}^{n}(A_k)_{jl}\boldsymbol{x}_l + \sum_{l=1}^{n_u}(B_k)_{jl}\boldsymbol{u}_l + \sum_{l=1}^{n}(K_i(\widehat{\boldsymbol{z}})C)_{jl}\boldsymbol{e})$$

which means that in order to obtain conditions similar to those of Theorems 3 and 4, $(TK_i(\widehat{\boldsymbol{z}})C)_{jl}$ has to bounded, leading to a bound on each entry of the matrices $TP_{\widehat{\boldsymbol{z}}}^{-1}L_iC$, while both $P_{\widehat{\boldsymbol{z}}}$ and L_i, $i = 1, 2, \ldots, r$ have to be designed.

A way to overcome this problem is to consider the observer

$$\begin{aligned}\dot{\widehat{\boldsymbol{x}}} &= \sum_{i=1}^{r} h_i(\widehat{\boldsymbol{z}})(A_i\widehat{\boldsymbol{x}} + B_i\boldsymbol{u} + L_i(\boldsymbol{y} - \widehat{\boldsymbol{y}}))\\ \widehat{\boldsymbol{y}} &= C\widehat{\boldsymbol{x}}\end{aligned} \tag{21}$$

with L_i, $i = 1, 2, \ldots, r$ constant matrices. The error dynamics are similar to (20):

$$\begin{aligned}\dot{\boldsymbol{e}} &= \sum_{i=1}^{r} h_i(\boldsymbol{z})(A_i\boldsymbol{e} - L_iC\boldsymbol{e})\\ &\quad + \sum_{i=1}^{r}(h_i(\boldsymbol{z}) - h_i(\widehat{\boldsymbol{z}}))(A_i\widehat{\boldsymbol{x}} + B_i\boldsymbol{u} + L_iC\boldsymbol{e})\end{aligned} \tag{22}$$

Now one can use the Lyapunov function $V = \boldsymbol{e}^{\mathrm{T}}\sum_{i=1}^{r} h_i(\boldsymbol{z})P_i\boldsymbol{e} = \boldsymbol{e}^{\mathrm{T}}P_{\boldsymbol{z}}\boldsymbol{e}$, with $P_i = P_i^{\mathrm{T}} > 0$, $i = 1, 2, \ldots, r$. Then,

$$\begin{aligned}\dot{V} =& \boldsymbol{e}^{\mathrm{T}}\sum_{i=1}^{r} h_i(\boldsymbol{z})\mathcal{H}(P_{\boldsymbol{z}}(A_i - L_iC))\boldsymbol{e} + \boldsymbol{e}^{\mathrm{T}}\dot{P}_{\boldsymbol{z}}\boldsymbol{e}\\ &+ 2\boldsymbol{e}^{\mathrm{T}}\sum_{i=1}^{r} P_{\boldsymbol{z}}(h_i(\boldsymbol{z}) - h_i(\widehat{\boldsymbol{z}}))(A_i\widehat{\boldsymbol{x}} + B_i\boldsymbol{u} + L_iC\boldsymbol{e}).\end{aligned}$$

[2]Since the weighting functions are assumed to be continuous, and due to the fact that the variables are defined on a compact set, $(h_i(\boldsymbol{z}) - h_i(\widehat{\boldsymbol{z}}))(A_i\boldsymbol{x} + B_i\boldsymbol{u}))$ is Lipschitz continuous in $\boldsymbol{z} - \widehat{\boldsymbol{z}}$, and, under mild conditions on $\boldsymbol{z}$, in the error $\boldsymbol{e}$.

With a similar assumption[3] as above, that $\sum_{i=1}^{r}(h_i(\boldsymbol{z}) - h_i(\widehat{\boldsymbol{z}}))(A_i\widehat{\boldsymbol{x}} + B_i\boldsymbol{u} + L_iC\boldsymbol{e}) = \Delta\boldsymbol{e}$, with $\|\Delta\| \leq \lambda$, we have

$$\begin{aligned}\dot{V}_c =& \boldsymbol{e}^{\mathrm{T}}\sum_{i=1}^{r} h_i(\boldsymbol{z})\mathcal{H}(P_{\boldsymbol{z}}(A_i - L_iC)) + \boldsymbol{e}^{\mathrm{T}}\sum_{i=1}^{r}\mathcal{H}(P_{\boldsymbol{z}}\Delta)\boldsymbol{e}\\ \leq& \boldsymbol{e}^{\mathrm{T}}\sum_{i=1}^{r} h_i(\boldsymbol{z})\mathcal{H}(P_{\boldsymbol{z}}(A_i - L_iC + \Delta))\boldsymbol{e}\\ \leq& \boldsymbol{e}^{\mathrm{T}}\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\boldsymbol{z})h_j(\boldsymbol{z})\mathcal{H}(P_jA_i - P_jL_iC + P_j\Delta))\boldsymbol{e}\end{aligned}$$

in which case $\dot{V}_c < 0$ can be formulated as an BMI problem.

Now, $\dot{P}_{\boldsymbol{z}}$ is simply

$$\dot{P}_{\boldsymbol{z}} = \sum_{i=1}^{r}\sum_{j=1}^{p} h_i(\boldsymbol{z})\frac{\partial w_0^j}{\partial \boldsymbol{z}_j}(P_{g_1(i,j)} - P_{g_2(i,j)})(T\dot{\boldsymbol{x}})_j$$

and for $\dot{\boldsymbol{x}}_j$ we have (similarly to the previous section)

$$\dot{\boldsymbol{x}}_j = \sum_{k=1}^{r} h_k(\boldsymbol{x})(\sum_{l=1}^{n}(A_k)_{jl}\boldsymbol{x}_l + \sum_{l=1}^{n_u}(B_k)_{jl}\boldsymbol{u}_l)$$

Then,

$$\begin{aligned}\dot{P}_{\boldsymbol{z}} =& \sum_{i=1}^{r}\sum_{j=1}^{p} h_i(\boldsymbol{z})\sum_{k=1}^{r} h_k(\boldsymbol{z})\Big[\sum_{l=1}^{n}\frac{\partial w_0^j}{\partial \boldsymbol{z}_j}\boldsymbol{x}_l(TA_k)_{jl}\\ &+ \sum_{l=1}^{n_u}\frac{\partial w_0^j}{\partial \boldsymbol{z}_j}\boldsymbol{u}_l(TB_k)_{jl}\Big](P_{g_1(i,j)} - P_{g_2(i,j)})\end{aligned}$$

Denote $\frac{\partial w_0^j}{\partial z_j}\boldsymbol{x}_l$, $l = 1, \ldots, n$, and $\frac{\partial w_0^j}{\partial z_j}\boldsymbol{u}_l$, $l = 1, \ldots, n_u$ by q_{jl}, $l = 1, \ldots, n + n_u$ and $(TA_k)_{jl}$, $l = 1, \ldots, n$, $(TB_k)_{jl}$, $l = 1, \ldots, n_u$ by M_{jl}, $l = 1, \ldots, n + n_u$, respectively. Then,

$$\dot{P}_{\boldsymbol{z}} = \sum_{i=1}^{r}\sum_{j=1}^{p} h_i(\boldsymbol{z})\sum_{k=1}^{r} h_k(\boldsymbol{z})\sum_{l=1}^{n+n_u} q_{jl}M_{jl}(P_{g_1(i,j)} - P_{g_2(i,j)})$$

The derivative can now be stated as

$$\begin{aligned}\dot{V} =& \boldsymbol{e}^{\mathrm{T}}\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\boldsymbol{z})h_j(\boldsymbol{z})\mathcal{H}(P_jA_i - P_jL_iC + P_j\Delta_i))\boldsymbol{e}\\ &+ \boldsymbol{e}^{\mathrm{T}}\sum_{i=1}^{r}\sum_{j=1}^{p} h_i(\boldsymbol{z})\sum_{k=1}^{r} h_k(\boldsymbol{z})\sum_{l=1}^{n+n_u} q_{jl}M_{jl}(P_{g_1(i,j)} - P_{g_2(i,j)})\boldsymbol{e}.\end{aligned}$$

Knowing the bounds $|q_{jl}| < \lambda_{jl}$, $l = 1, \ldots, n + n_u$, the design problem can be formulated as a BMI problem, as follows:

Corollary 1: If there exists matrices $P_i = P_i^{\mathrm{T}} > 0$, and L_i, $i = 1, 2, \ldots, r$ such that

$$\begin{aligned}&\Upsilon_{ii}^m < 0 \qquad i = 1, 2, \ldots, r,\ m = 1, \ldots, 2^{p\times(n+n_u)}\\ &\frac{2}{r-1}\Upsilon_{ii}^m + \Upsilon_{ij}^m + \Upsilon_{ji}^m < 0\\ &i, j = 1, 2, \ldots, r,\ i \neq j,\ m = 1, \ldots, 2^{p\times(n+n_u)}\end{aligned} \tag{23}$$

with

$$\Upsilon_{ij}^m = \begin{pmatrix} G_{ij} & P_i \\ P_i & -2S_j \end{pmatrix} \tag{24}$$

with $G_{ij} = P_iA_j - P_iL_jC + A_j^{\mathrm{T}}P_i - C^{\mathrm{T}}L_j^{\mathrm{T}}P_i + \frac{1}{2}\lambda_j^2S_j + \sum_{k=1}^{p}\sum_{l=1}^{n+2n_u}(-1)^{d_{kl}^m}\lambda_{kl}M_{kl}(P_{g_1(i,k)} - P_{g_2(i,k)})$ *hold, where d_{kl}^m is defined from the binary representation of $m-1$ and $g_1(i,k)$, $g_2(i,k)$ are defined as in* (12)*, then the estimation error $\boldsymbol{e}$ of the observer* (21) *tends to zero exponentially for any trajectory of $\boldsymbol{x}$ satisfying* (3) *in the outermost Lyapunov level contained in $\bigcap_{k,l}\{(\boldsymbol{x}, \boldsymbol{u}) : |q_{kl}| \leq \lambda_{kl}\}$.*

Proof: Follows directly from applying Theorem 1 to the derivative of the Lyapunov function.

A similar result can also be formulated based on Theorem 4.

Remark: Although this approach presents a way to design local observers also in the case when the scheduling vector depends on states to be estimated, one has to note that solving a BMI problem is much harder than solving an LMI problem.

IV. Examples

In this section, we illustrate the proposed observer design on two simulation examples.

A. Measured scheduling vector

Consider a nonlinear system given as

$$\begin{aligned}\dot{\boldsymbol{x}} =& \begin{pmatrix} 3 & 7\boldsymbol{x}_1^2 + 3 & 8 \\ 4 & 3 & 5 \\ 7 & 2 & 7 + \sqrt[3]{\frac{1-\boldsymbol{x}_1}{2}} \end{pmatrix}\boldsymbol{x} + \begin{pmatrix} x_1^2 + 2 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}\boldsymbol{u}\\ \boldsymbol{y} =& \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}\boldsymbol{x}\end{aligned} \tag{25}$$

with $\boldsymbol{x}_i, \boldsymbol{u}_i \in [-1, 1]$, $i = 1, 2$. Our goal is to design an observer for this system. Using the sector nonlinearity approach, a four-rule TS fuzzy system can be constructed, with

$$\begin{aligned}\dot{\boldsymbol{x}} =& \sum_{i=1}^{4} h_i(\boldsymbol{z})(A_i\boldsymbol{x} + B_i\boldsymbol{u})\\ \boldsymbol{y} =& C\boldsymbol{x}\end{aligned} \tag{26}$$

$A_1 = \begin{pmatrix} 3 & 10 & 8 \\ 4 & 3 & 5 \\ 7 & 2 & 7 \end{pmatrix}$, $A_2 = \begin{pmatrix} 3 & 10 & 8 \\ 4 & 3 & 5 \\ 7 & 2 & 8 \end{pmatrix}$, $A_3 = \begin{pmatrix} 3 & 3 & 8 \\ 4 & 3 & 5 \\ 7 & 2 & 7 \end{pmatrix}$, $A_4 = \begin{pmatrix} 3 & 3 & 8 \\ 4 & 3 & 5 \\ 7 & 2 & 8 \end{pmatrix}$, $\boldsymbol{z} = [\boldsymbol{x}_1, \boldsymbol{x}_1]^{\mathrm{T}} = [\boldsymbol{y}, \boldsymbol{y}]^{\mathrm{T}}$, $w_0^1 = 1 - \boldsymbol{x}_1^2$, $w_1^1 = 1 - w_0^1$, $w_0^2 = 1 - \sqrt[3]{\frac{1-\boldsymbol{x}_1}{2}}$, $w_1^2 = 1 - w_0^2$, $h_1 = w_0^1 \cdot w_0^2$, $h_2 = w_0^1 \cdot w_1^2$, $h_3 = w_1^1 \cdot w_0^2$, $h_4 = w_1^1 \cdot w_1^2$, $B_1 = B_2 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$, $B_3 = B_4 = \begin{pmatrix} 3 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$, $C = [1\ 0\ 0]^{\mathrm{T}}$, $T_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $T_2 = 0$. Note that the local matrices are unstable, and an observer cannot be designed using a common quadratic Lyapunov function. However, there exist

[3]Although λ exists, in practical situations it is hard to determine it, as it depends on the observer gains. In order to solve the problem, however, it is possible to bound the observer gains and compute the maximum λ for which the BMI problem is feasible.

P_i, L_i, $i = 1, 2, \ldots, r$ such that the LMIs $P_z A_z - L_z C < 0$ are feasible. With these weighting functions we have $\frac{\partial w_0^1}{\partial \boldsymbol{z}_1} = -2\boldsymbol{x}_1$ and $\frac{\partial w_0^2}{\partial \boldsymbol{z}_1} = \frac{1}{\sqrt[3]{2(1-\boldsymbol{x}_1)^2}}$. The $\lambda_{\max}$ that can be obtained using Theorem 4 is $\lambda_{\max} = 1.2256$, that leads to the conditions:

$$\begin{aligned}
&|\boldsymbol{x}_1| \leq 0.783 \\
&|\boldsymbol{x}_1 \boldsymbol{x}_i| \leq 0.61 \\
&|\boldsymbol{x}_1 \boldsymbol{u}_i| \leq 0.61 \quad i = 1, 2 \\
&\left|\frac{\boldsymbol{x}_i^3}{\sqrt[3]{(1-\boldsymbol{x}_1)^2}}\right| \leq 1.2256\sqrt[3]{2} \quad i = 1, 2 \\
&\left|\frac{\boldsymbol{u}_i^3}{\sqrt[3]{(1-\boldsymbol{x}_1)^2}}\right| \leq 1.2256\sqrt[3]{2} \quad i = 1, 2
\end{aligned} \tag{27}$$

The estimation error for a trajectory of the states that obeys the above conditions is presented in Figure 1. The true initial conditions were $[0.5\ 1\ 1]^{\mathrm{T}}$, while the estimated ones were $[0\ 0\ 0]^{\mathrm{T}}$.

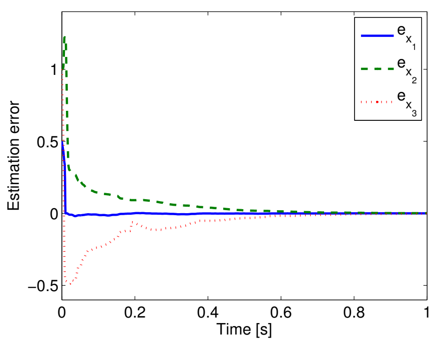

Fig. 1. Estimation error.

B. State-dependent scheduling vector

Consider now a two-rule fuzzy system with $A_1 = \begin{pmatrix} -2 & 1 \\ -4 & -3 \end{pmatrix}$, $A_2 = \begin{pmatrix} -2 & -1 \\ -4 & 2 \end{pmatrix}$, $C = [1, 0]$, $h_1(\boldsymbol{x}_2) = w_1(\boldsymbol{x}_2) = \frac{1}{2}(1 + \cos \boldsymbol{x}_2)$, $\boldsymbol{x}_1, \boldsymbol{x}_2 \in [-1, 1]$. Note that the second local model is unstable.

As can be seen, the membership functions depend on $\boldsymbol{x}_2$, which has to be estimated. We have

$$\begin{aligned}
|h_1(\boldsymbol{x}_2) - h_1(\widehat{\boldsymbol{x}}_2)| &= \frac{1}{2}\left|\cos \boldsymbol{x}_2 - \cos \widehat{\boldsymbol{x}}_2\right| \\
&= \left|\sin \frac{\boldsymbol{x}_2 + \widehat{\boldsymbol{x}}_2}{2} \sin \frac{\boldsymbol{x}_2 - \widehat{\boldsymbol{x}}_2}{2}\right| \leq \frac{\|\boldsymbol{e}\|}{2}
\end{aligned}$$

for $\boldsymbol{x}_1, \boldsymbol{x}_2 \in [-1, 1]$. Moreover, since the states are bounded, $\|A_i \boldsymbol{x}\| \leq 7.25$. With these values, using a common quadratic Lyapunov function, the LMIs used to design the observer are not strictly feasible.

However, there exist P_i, L_i, $i = 1, 2, \ldots, r$ such that the BMIs $P_z A_z - P_z L_z C < 0$ are feasible. The BMIs have been solved using *Penbmi* [7], maximizing λ and λ_{ij} and minimizing $\|L_i C\|$ simultaneously. Using the method in Section III-B, we obtain $\lambda = 0.3$ and $\max(\lambda_{ij}) = 0.28$, which lead to the following conditions:

$$\begin{aligned}
&|\boldsymbol{x}_1 \sin \boldsymbol{x}_2| \leq 0.07 \\
&|\boldsymbol{x}_2 \sin \boldsymbol{x}_2| \leq 0.093 \\
&\|\boldsymbol{e}\| \leq 0.075
\end{aligned} \tag{28}$$

i.e., the estimated states converge to the true ones as long as conditions (28) are satisfied.

V. CONCLUSIONS

In this paper we presented a local observer design method for TS fuzzy systems obtained by the sector nonlinearity approach. The observer is valid locally in a domain that is determined by solving an LMI or a BMI problem. As the examples illustrate, although the observer is only local, the design conditions are less conservative than classical design conditions.

In this paper we only considered the case when the measurement matrix is common for all the rules of the system, and the input is also a measured variable. The case when the measurement matrices are different will be investigated in our future research.

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