# Stabilization of time-delay nonlinear systems using Takagi-Sugeno fuzzy models 

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#### Abstract

We propose a controller design method for time-delay nonlinear systems with delays affecting both the states and the inputs, represented by Takagi-Sugeno fuzzy models with nonlinear consequents. To handle the nonlinearities in the consequents we assume that they are slope-bounded. Linear matrix inequality conditions are formulated to design the controller. The obtained results are compared to state-of-the-art approaches and illustrated on two examples.


Keywords: time-delay systems, stabilization, T-S fuzzy systems, linear matrix inequalities

## 1. Introduction

Stability and control of systems that involve physical time lag have attracted an increasing interest in the past years. In many applications, when the actuators and sensors are not in the same place, time-delay appears. For example in networked control systems, transmission delay [1], network induced delays [2], and delay in finite switching speed of amplifiers [3] have to be considered. Delays can also appear in transportation [4], in biological systems [5], or input delay in multiagent systems [6, 7]. In active suspension systems [8] the time delay is due to the hydraulic or pneumatic actuation. A system's performance may be degraded because of the delay, thus it is important to consider it in the analysis and the design.

Stabilization of time-delay systems has a long tradition and there have been extensive contributions, see e.g. [9, 10, 11] related to stability and control in the presence of delays of linear and nonlinear systems. Usually Lyapunov's direct method is used for analysis and design.

In this paper we study stabilization of delayed nonlinear systems, where both the states and the input are affected by delay. We represent the nonlinear system by Takagi-Sugeno (TS) models with nonlinear consequents. Many nonlinear time-delayed systems can be represented by T-S systems, which are a convex combination of local models.

T-S models with time-delay have obtained significant consideration in the last years. For instance, stability conditions based on an augmented Lyapunov-Krasovskii functional have been developed in [12]. [13] used a PDC controller for robust stabilization. Enhanced stability and stabilization conditions that depend on the delay have been presented in [14]. An adaptive fuzzy finite-time control scheme to guarantee the stability of nonlinear power systems with actuator faults and time delays has been presented in [15]. Conditions for stabilization of T-S models are generally formulated as Linear Matrix Inequalities (LMIs).

T-S models obtained by the approach in [16] have the drawback of the potentially large number of rules. Leaving some nonlinearities in their original form reduces the number of models. The separation of the nonlinearities has been exploited in several results for both continuous [17, 18, 19, 20] and discrete-time case [21]. The nonlinearities are generally handled by a sectorbound condition [17, 20, 21] or incremental quadratic constraints [18, 19]. Unlike existing results, in this paper we adopt a slope-bound condition [22, 23], that allows considering a different class of nonlinearities.

Regarding the variables affected by the delay, [24, 25, 26, 27, 28] consider delay in the states, while in [8] delay in the input is considered. [29] has investigated T-S systems with nonlinear consequents, when both the states and the input may depend on delayed variables. In this paper we allow the membership functions to depend on the delayed variables, a case that is barely treated in the literature. Although in general different delays may affect the distinct states and inputs, for simplicity, in this paper we consider the same delay. One problem that has to be emphasized is that if the control law is nonlinear, with state-dependent controller gains, the gains will also be affected by the delay.

The delay is supposed to be unknown, but bounded and with bounded derivative, with known bounds. Note that this does not render the control law non-implementable, as the delay is inherent to the system.

In our previous work, we considered a simple Lyapunov functional that depends on the derivative of the delay in [30] and a delay and its derivativedependent Lyapunov functional in [31], where all the terms involved constant
matrices. In this paper we present a general solution, based on a non-PDC control law and a fuzzy delay-dependent Lyapunov functional. After presenting generic conditions, we analyse specific cases for which sufficient design conditions can be formulated as LMIs. As far as we know, no previous research has investigated such a general stabilization condition for T-S fuzzy systems with nonlinear consequents.

The structure of this paper is as follows: in Section 2 we present the necessary background, assumptions used, and problem statement. The conditions for stabilization are developed in Section 3. They are compared to state-ofthe art results and are illustrated on two examples in Section 4. Section 5 concludes the paper.

Notation. The notation we use is standard. For a real symmetric matrix $F=F^{T} \in R^{n \times n}, F$ being positive (negative) definite is denoted by $F>0$ $(F<0)$. The identity matrix is denoted by $I$ and the zero matrix by 0 . The symbol $*$ is used for the symmetric element, e.g., $\left(\begin{array}{cc}P & A \\ * & P\end{array}\right)=\left(\begin{array}{cc}P & A \\ A^{T} & P\end{array}\right)$, and $A+*=A+A^{T} . \operatorname{diag}\left(f_{1}, \ldots, f_{n}\right)$ denotes the block-diagonal matrix having $f_{1}, \ldots, f_{n}$ on the diagonal. $\|x\|$, where $x \in \mathbb{R}^{n_{x}}$, is the Euclidean norm of $x$. For simplicity, convex sums of matrix expressions and the time they are evaluated at are indicated by the subscript, e.g.,

$$
\begin{equation*}
F_{z z_{\tau}}=\sum_{i=1}^{r} q_{i}(z(t)) \sum_{j=1}^{r} q_{j}(z(t-\tau(t))) F_{i j} \tag{1}
\end{equation*}
$$

where $q_{i}, i=1, \ldots, r$ are the normalized membership functions:

$$
\begin{equation*}
q_{i} \in[0,1], i=1, \ldots, r, \quad \sum_{i=1}^{r} q_{i}(t)=1, \forall t \tag{2}
\end{equation*}
$$

Furthermore, we use the notation $F(z)$ in general for matrix expressions that depend (possibly nonlinearly) on the time-varying variable $z(t)$.

## 2. Preliminaries and problem statement

We consider nonlinear systems of the form

$$
\begin{align*}
\dot{x}(t)= & A(z(t), z(t-\tau(t))) x(t) \\
& +D(z(t), z(t-\tau(t))) x(t-\tau(t)) \\
& +B(z(t), z(t-\tau(t))) u(t-\tau(t))  \tag{3}\\
& +B(z(t), z(t-\tau(t))) G \psi(H x(t)),
\end{align*}
$$

where $z$ is a variable that may depend on the system's states, outputs or other measured variables, $x(t) \in \mathbb{R}^{n_{x}}$ is the state vector, $u(t) \in \mathbb{R}^{n_{u}}$ is the control input, and $\tau(t)$ is the varying time-delay. The matrix $G \in \mathbb{R}^{n_{u} \times p}$ is needed in order to obtain the same dimensions with the input. The nonlinearities in the matrices $A \in \mathbb{R}^{n_{x} \times n_{x}}, D \in \mathbb{R}^{n_{x} \times n_{x}}$, and $B \in \mathbb{R}^{n_{x} \times n_{u}}$ are handled with T-S fuzzy representation. The nonlinearities in the vector $\psi(\cdot)$ are handled using slope-bound conditions.

A somewhat restrictive assumption we make on the nonlinear system is the match between the input and the nonlinearity $\psi(\cdot)$. The motivation of this consists in the way how the models are usually obtained. For instance, the dynamic model of a robot arm has the following nonlinear dynamics:

$$
M(q) \ddot{q}=-C(q, \dot{q}) \dot{q}-G(q)+\tau,
$$

where $q$ is the angle, $\dot{q}$ is the angular velocity, $\tau$ is the torque, $M(q)$ is the mass matrix, $C(q, \dot{q})$ contains the Coriolis and the centrifugal matrices, and $G(q)$ is the gravity matrix. To use this model in the classical form, we need to multiply with the inverse of the mass matrix, leading to:

$$
\ddot{q}=-M(q)^{-1} C(q, \dot{q}) \dot{q}-M(q)^{-1} G(q)+M(q)^{-1} \tau .
$$

In this example $B$ is $M(q)^{-1}$, a matrix that also multiplies any nonlinearity that appears. The multiplication with the inverse of the mass matrix often appears for models obtained from first principles, therefore this motivates the form of the model in (3).

The nonlinear system in (3) has a general form where the input, the system states, and $z$ are affected by the delay. Special cases can be derived from this form. For example, when the delay only affects the input, $D=0$, or the nonlinearities in the matrices may depend only on the current value of $z(t)$.

The term $\psi(H x(t)) \in \mathbb{R}^{p}, H \in \mathbb{R}^{p \times n_{x}}$ is a vector function where each element is a function of a linear combination of the states, i.e.

$$
\psi_{i}=\psi_{i}\left(\sum_{j=1}^{n_{x}} H_{i j} x_{j}(t)\right), \quad i=1, \ldots, p
$$

We assume that each entry in $\psi(H x(t))$ satisfies:

Assumption 1. [32] For any $i \in\{1, \ldots, p\}$ there exist constants $0<b_{i} \leq$ $\infty$, so that

$$
\begin{equation*}
0 \leq \frac{\psi_{i}(v)-\psi_{i}(w)}{v-w} \leq b_{i}, \quad \forall v, w \in \mathbb{R}, v \neq w \tag{4}
\end{equation*}
$$

Thus, see [33], there exist $\delta_{i}(t) \in\left[0, b_{i}\right]$, so that for any $v, w \in \mathbb{R}$

$$
\begin{equation*}
\psi_{i}(v)-\psi_{i}(w)=\delta_{i}(t)(v-w) \tag{5}
\end{equation*}
$$

We use the notation $\delta(t)=\operatorname{diag}\left(\delta_{1}(t), \ldots, \delta_{p}(t)\right)$.
We represent system (3) by the time-delay T-S fuzzy model with nonlinear consequents:

$$
\begin{align*}
\dot{x}(t)= & A_{z z_{\tau}} x(t)+D_{z z_{\tau}} x(t-\tau(t)) \\
& +B_{z z_{\tau}} u(t-\tau(t))+B_{z z_{\tau}} G \psi(H x(t)), \tag{6}
\end{align*}
$$

where $A_{z z_{\tau}}, D_{z z_{\tau}}$, and $B_{z z_{\tau}}$ are the model matrices, having the form:

$$
A_{z z_{\tau}}=\sum_{i=1}^{r} \sum_{j=1}^{r} q_{i}(z(t)) q_{j}(z(t-\tau(t))) A_{i j} x(t)
$$

with $r$ being the number of rules and $q_{i}, i=1, \ldots, r$ are membership functions that satisfy condition (2).

In the remainder of this paper, for simplicity, we denote $\tau(t)$ by $\tau . \tau$ is assumed to be differentiable, $\dot{\tau} \leq d, d \in[0,1)$ and bounded, $\tau \leq h$, where $d$ and $h>0$ are known constants.

One of the main novelties of this work consists in the way how the nonlinearities in the vector $\psi(\cdot)$ are handled. For similar stabilization problems in the literature the sector-bound condition is used, see e.g. [29, 34, 35], which is defined as:

$$
\psi(x) \in \operatorname{co}\{0, E x\} .
$$

The notation co refers to the convex hull and Ex defines a linear combination of $x$. In contrast to this, in this paper the nonlinear consequents are handled using a condition on the slope of the nonlinearity. In what follows we provide an example to highlight the main differences between these two conditions.

Example 1. Consider the nonlinear function:

$$
\psi(v)=v+\sin (v)
$$

The sector-bounds are $0 v$ and $2 v$, and the nonlinear functions with the sector bounds can be seen in Fig. 1(a). Intuitively, this can be interpreted as the nonlinear function being in the sector delimited by $0 v$ and $2 v$.

On the other hand, the slope-bound condition defines a bound on the slope of the nonlinearity. In the case of continuous, differentiable functions, these limits are equal to the minimum and maximum of the derivatives, see e.g. [36]. For this example the slope-bounds are 0 and 2 and the nonlinear function with the slope-bounds can be seen in Fig. 1(b).

(a) $\psi(v)$ with sector-bounds

(b) $\psi(v)$ with minimum and maximum slope

Figure 1: Sector and slope bound comparisons

We can see that there are fundamental differences between the two conditions, and none of them includes the other. The sector-bound condition does not allow nonlinearities that contain affine terms or that are not zero at the origin. The slope-bound condition defined in (4) makes it possible to also include such functions, e.g. $\psi(v)=v+\cos (v)$.

To complete our problem statement we present the following property and lemmas that are necessary to develop our results.

Lemma 1 ([31]). Let $A$ and $B$ be matrices of appropriate dimensions and ranks, with $B=B^{T}>0$. Then

$$
-A^{T} B^{-1} A \leq-A-A^{T}+B
$$

Property 1 (Schur complement [37]). Let $\mathcal{M}=\mathcal{M}^{T}=\left[\begin{array}{ll}M_{11} & M_{12} \\ M_{12}^{T} & M_{22}\end{array}\right]$, with $M_{11}$ and $M_{22}$ square matrices of appropriate dimensions. Then:

$$
\begin{align*}
\mathcal{M}<0 & \Leftrightarrow\left\{\begin{array}{l}
M_{11}<0 \\
M_{22}-M_{12}^{T} M_{11}^{-1} M_{12}<0
\end{array}\right. \\
& \Leftrightarrow\left\{\begin{array}{l}
M_{22}<0 \\
M_{11}-M_{12} M_{22}^{-1} M_{12}^{T}<0
\end{array}\right. \tag{7}
\end{align*}
$$

Sufficient LMI conditions for the multiple sum negativity problem containing the double sum

$$
\begin{equation*}
\sum_{i=1}^{r} \sum_{j=1}^{r} q_{i}(z(s)) q_{j}(z(s)) F_{i j}<0 \tag{8}
\end{equation*}
$$

with symmetric matrices $F_{i j}$, and membership functions $q_{i}, i=1, \ldots, r$, will be obtained by the following lemma:

Lemma 2 ([38]). Equation (8) is satisfied if the following condition holds

$$
\begin{equation*}
F_{i j}+F_{j i}<0 \quad \forall i, j=1, \ldots, r, i \geq j \tag{9}
\end{equation*}
$$

Remark 1. Using more advanced relaxations on (26), e.g. [39, 40, 41], less conservative results can be obtained. For simplicity we use Lemma 2.

We design a state-feedback controller under the assumption that all the states are available. Our objective is to develop sufficient conditions such that the controller stabilizes system (6). We use the control law [31]:

$$
\begin{equation*}
u(t)=-K(z) x(t)-G \psi(H x(t)) \tag{10}
\end{equation*}
$$

where $K(z)$ contains the controller gains. Combining (6) and (10), using Assumption 1 on $\psi(H x(t))-\psi(H x(t-\tau))$, and denoting $\eta:=H(x(t)-x(t-\tau))$, we get

$$
\begin{align*}
\dot{x}(t) & =A_{z z_{\tau}} x(t)+\left(D_{z z_{\tau}}-B_{z z_{\tau}} K(z(t-\tau))\right) x(t-\tau)+B_{z z_{\tau}} G \delta(t) \eta  \tag{11}\\
\eta & =H(x(t)-x(t-\tau))
\end{align*}
$$

Remark 2. Recall that we assume $\psi(\cdot)$ is known, and we use this term in its original form to reduce the number of local models. If there is no delay, $\psi(\cdot)$ is cancelled in the closed-loop system, i.e., $\eta=0$. However, if there is a delay in the input, the difference $\psi(H x(t))-\psi(H x(t-\tau))$ appears, as shown above. We use Assumption 1 to handle the difference, this being one of the main contributions of this work.

## 3. Main results

In what follows we first formulate general conditions for the stabilization of system (11) and then discuss special cases for which we develop sufficient LMI conditions.

### 3.1. General conditions

The Lyapunov-Krasovskii functional we use is:

$$
\begin{align*}
V(t, x, \dot{x}) & =x^{T}(t) P_{z}^{-1} x(t) \\
& +\int_{t-h}^{t} x^{T}(s) P(s)^{-1} S(s) P(s)^{-1} x(s) d s \\
& +h \int_{-h}^{0} \int_{t+\theta}^{t} \dot{x}^{T}(s) P(s)^{-1} R(s) P(s)^{-1} \dot{x}(s) d s d \theta  \tag{12}\\
& +\int_{t-\tau}^{t} x^{T}(s) P(s)^{-1} Q(s) P(s)^{-1} x(s) d s
\end{align*}
$$

where $P(s)=\sum_{i=1}^{r} q_{i}(z(s)) P_{i}, S(s)=\sum_{i=1}^{r} q_{i}(z(s)) Q_{i}$, $R(s)=\sum_{i=1}^{r} q_{i}(z(s)) R_{i}, Q(s)=\sum_{i=1}^{r} q_{i}(z(s)) Q_{i}$.

Note that in this paper we generalize our previous work presented in [30, 31]. The introduction of fuzzy matrices in (12) relaxes the design conditions. The special cases in $[30,31]$ can be recovered as:

- The results of [30] can be found by taking $S(s)=0, R(s)=0$, and constant matrices $P_{z}^{-1}=P, P(s)^{-1} Q(s) P(s)^{-1}=\tilde{Q}$ in (12).
- The results of [31] can be recovered by taking constant matrices:
$P(s)^{-1} S(s) P(s)^{-1}=\tilde{S}, P(s)^{-1} R(s) P(s)^{-1}=\tilde{R}, P_{z}^{-1}=P$, $P(s)^{-1} Q(s) P(s)^{-1}=\tilde{Q}$ in (12).

General conditions are formulated as:
Theorem 1. Consider the closed-loop system (11) and assume that $\dot{\tau} \leq d$, $d \in[0,1), \tau \leq h, h>0$, and there exists a domain $\mathcal{D}$ with $0 \in \mathcal{D}$, such that $\phi_{i} \leq \dot{q}_{i}(z), i=1, \ldots, r, \forall z \in \mathcal{D}$. The closed-loop system (11), with $d$, $h$, and $\phi_{i}, i=1, \ldots, r$, being known constants, is locally asymptotically stable in the largest Lyapunov level-set included in $\mathcal{D}$, if there exist $P_{i}=P_{i}^{T}>0$, $R_{i}=R_{i}^{T}>0, \tilde{R}=\tilde{R}^{T}>0,\left[\begin{array}{cc}R_{i} & P_{i} \\ P_{i} & \tilde{R}\end{array}\right] \geq 0, S_{12},\left[\begin{array}{cc}\tilde{R}^{-1} & S_{12} \\ S_{12}^{T} & \tilde{R}^{-1}\end{array}\right] \geq 0, S_{i}=S_{i}^{T}>$ $0, Q_{i}=Q_{i}^{T}>0, X=X^{T}, P_{i}+X \geq 0, M=\operatorname{diag}\left(m_{1}, \ldots, m_{p}\right)>0, N_{i}$, $i=1, \ldots, r$, and $\epsilon>0$, so that

$$
\left[\begin{array}{ccccc}
\mathcal{E}_{11}^{13} & P_{z} S_{12} P_{z_{h}} & \mathcal{E}_{13}^{13} & B_{z z_{\tau}} G M^{-1}+P_{z} H^{T} & h P_{z} A_{z z_{\tau}}^{T}  \tag{13}\\
* & \mathcal{E}_{22}^{13} & \mathcal{E}_{23}^{13} & 0 & 0 \\
* & * & \mathcal{E}_{33}^{13} & -P_{z_{\tau}} H^{T} & h P_{z_{\tau}} \mathcal{B}^{T} \\
* & * & * & M^{-1} \nu(M) M^{-1} & h M^{-1}\left(B_{z z_{\tau}} G\right)^{T} \\
* & * & * & * & -P_{z} R_{z}^{-1} P_{z}
\end{array}\right] \leq 0
$$

holds, with

$$
\begin{align*}
\mathcal{B} & =D_{z z_{\tau}}-B_{z z_{\tau}} K(z(t-\tau)), \\
\mathcal{E}_{11}^{13} & =P_{z} A_{z z_{\tau}}^{T}+A_{z z_{\tau}} P-P_{\phi}+\epsilon P_{z}^{2}+S_{z}+Q_{z}-P_{z} \tilde{R}^{-1} P_{z}, \\
\mathcal{E}_{13}^{13} & =\mathcal{B} P_{z_{\tau}}+P_{z} \tilde{R}^{-1} P_{z_{\tau}}-P_{z} S_{12} P_{z_{\tau}}, \\
\mathcal{E}_{22}^{13} & =-P_{z_{h}} \tilde{R}^{-1} P_{z_{h}}-S_{z_{h}} \\
\mathcal{E}_{23}^{13} & =P_{z_{h}} \tilde{R}^{-1} P_{z_{\tau}}-P_{z_{h}} S_{12}^{T} P_{z_{\tau}},  \tag{14}\\
\mathcal{E}_{33}^{13} & =-(1-d) Q_{z_{\tau}}+P_{z_{\tau}}\left(-2 \tilde{R}^{-1}+S_{12}+S_{12}^{T}\right) P_{z_{\tau}} \\
\nu(M) & =-2 M \operatorname{diag}\left(\frac{1}{b_{1}}, \ldots, \frac{1}{b_{r}}\right), \\
P_{\phi} & =\sum_{k=1}^{r} \phi_{k}\left(P_{k}+X\right) .
\end{align*}
$$

Remark 3. If the conditions $\phi_{i} \leq \dot{q}_{i}(\cdot), i=1, \ldots, r$, hold globally, then the closed-loop system (11) is also globally asymptotically stable.

Proof. Differentiating (12) along the trajectories of (11) gives

$$
\begin{align*}
\dot{V}(t, x, \dot{x})= & x^{T}(t) P_{z}^{-1} \dot{x}(t)+\dot{x}^{T}(t) P_{z}^{-1} x(t)+x^{T}(t) \dot{P}_{z}^{-1} x(t) \\
& +h^{2} \dot{x}^{T}(t) P_{z}^{-1} R_{z} P_{z}^{-1} \dot{x}(t)-h \int_{t-h}^{t} \dot{x}^{T}(s) P(s)^{-1} R(s) P(s)^{-1} \dot{x}(s) d s \\
& +x^{T}(t) P_{z}^{-1}\left[S_{z}+Q_{z}\right] P_{z}^{-1} x(t)-x^{T}(t-h) P_{z_{h}}^{-1} S_{z_{h}} P_{z_{h}}^{-1} x(t-h) \\
& -(1-\dot{\tau}(t)) x^{T}(t-\tau) P_{z_{\tau}}^{-1} Q_{z_{\tau}} P_{z_{\tau}}^{-1} x(t-\tau) \tag{15}
\end{align*}
$$

Note that $\dot{P}_{z}^{-1}=-P_{z}^{-1} \dot{P}_{z} P_{z}^{-1}$.
Next, consider the condition $\left[\begin{array}{cc}R_{i} & P_{i} \\ P_{i} & \tilde{R}\end{array}\right] \geq 0, i=1,2, \ldots, r$, implying $\left[\begin{array}{cc}R(s) & P(s) \\ P(s) & \tilde{R}\end{array}\right] \geq 0$, and thus $P(s)^{-1} R(s) P(s)^{-1} \geq \tilde{R}^{-1}$.

Thanks to reciprocal convexity [9], [42]

$$
\begin{align*}
& -h \int_{t-h}^{t} \dot{x}^{T}(s) P(s)^{-1} R(s) P(s)^{-1} \dot{x}(s) d s \leq \\
& -h \int_{t-h}^{t} \dot{x}^{T}(s) \tilde{R}^{-1} \dot{x}(s) d s \leq  \tag{16}\\
& -\left[\begin{array}{ll}
e_{1} \\
e_{2}
\end{array}\right]^{T}\left[\begin{array}{cc}
\tilde{R}^{-1} & S_{12} \\
S_{12}^{T} & \tilde{R}^{-1}
\end{array}\right]\left[\begin{array}{l}
e_{1} \\
e_{2}
\end{array}\right],
\end{align*}
$$

where $e_{1}=x(t)-x(t-\tau(t)), e_{2}=x(t-\tau(t))-x(t-h)$, for any $S_{12}$ so that $\left[\begin{array}{cc}\tilde{R}^{-1} & S_{12} \\ S_{12}^{T} & \tilde{R}^{-1}\end{array}\right] \geq 0$, see $[9]$.

Using the notation $\chi:=\left[\begin{array}{llll}x(t)^{T} & x(t-h)^{T} & x(t-\tau)^{T} & (\delta(t) \eta)^{T}\end{array}\right]^{T}$ and the assumption that $\dot{\tau}(t) \leq d$ we obtain $\dot{V}(t, x, \dot{x}) \leq \chi^{T} \Delta \chi$, where $\Delta$ is defined as

$$
\Delta=\left[\begin{array}{cccc}
\mathcal{E}_{11}^{17} & S_{12} & \mathcal{E}_{12}^{17}+h^{2} A_{z z_{\tau}}^{T} P_{z}^{-1} R_{z} P_{z}^{-1} \mathcal{B} & \left(P_{z}^{-1}+h^{2} A_{z z_{\tau}}^{T} P_{z}^{-1} R_{z} P_{z}^{-1}\right) \mathcal{C}  \tag{17}\\
* & \mathcal{E}_{22}^{17} & \tilde{R}^{-1}-S_{12}^{T} & 0 \\
* & * & \mathcal{E}_{33}^{17}+h^{2} \mathcal{B}^{T} P_{z}^{-1} R_{z} P_{z}^{-1} \mathcal{B} & h^{2} \mathcal{B}^{T} P_{z}^{-1} R_{z} P_{z}^{-1} \mathcal{C} \\
* & * & * & h^{2} \mathcal{C}^{T} P_{z}^{-1} R_{z} P_{z}^{-1} \mathcal{C}
\end{array}\right]
$$

$$
\begin{aligned}
\mathcal{B} & =D_{z z_{\tau}}-B_{z z_{\tau}} K(z(t-\tau)), \\
\mathcal{C} & =B_{z z_{\tau}} G, \\
\mathcal{E}_{11}^{17} & =A_{z z_{\tau}}^{T} P_{z}^{-1}+P_{z}^{-1} A_{z z_{\tau}}-P_{z}^{-1} \dot{P}_{z} P_{z}^{-1}+h^{2} A_{z z_{\tau}}^{T} P_{z}^{-1} R_{z} P_{z}^{-1} A_{z z_{\tau}} \\
& +P_{z}^{-1}\left(S_{z}+Q_{z}\right) P_{z}^{-1}-\tilde{R}^{-1}, \\
\mathcal{E}_{12}^{17} & =P_{z}^{-1} \mathcal{B}+\tilde{R}^{-1}-S_{12}, \\
\mathcal{E}_{22}^{17} & =-\tilde{R}^{-1}-P_{z_{h}}^{-1} S_{z_{h}} P_{z_{h}}^{-1}, \\
\mathcal{E}_{33}^{17} & =-(1-d) P_{z_{\tau}}^{-1} Q_{z_{\tau}} P_{z_{\tau}}^{-1}-2 \tilde{R}^{-1}+S_{12}+S_{12}^{T} .
\end{aligned}
$$

Next, consider $\chi^{T} \theta \chi$, with

$$
\theta=\left[\begin{array}{cccc}
\epsilon I & 0 & 0 & H^{T} M  \tag{18}\\
* & 0 & 0 & 0 \\
* & * & 0 & -H^{T} M \\
* & * & * & \nu(M)
\end{array}\right]
$$

and $M=\operatorname{diag}\left(m_{1}, \ldots, m_{p}\right)>0$.
Following the same reasoning as in [31], one can prove that

$$
\begin{equation*}
-\chi^{T} \theta \chi \leq-\epsilon\|x(t)\|^{2} \tag{19}
\end{equation*}
$$

Accordingly, if $\chi^{T} \Delta \chi+\chi^{T} \theta \chi<0$, then $\dot{V}<0$.
The matrix inequality $\Delta+\theta<0$ can be separated as in (20), where $\mathcal{E}_{11}^{20}=\mathcal{E}_{11}^{17}+\epsilon I, \mathcal{E}_{33}^{20}=\mathcal{E}_{33}^{17}, \mathcal{R}=P_{z}^{-1} R_{z} P_{z}^{-1}$. Applying the Schur complement on (20) we get (21) with $\mathcal{E}_{11}^{21}=\mathcal{E}_{11}^{20}, \mathcal{E}_{33}^{21}=\mathcal{E}_{33}^{20}, \mathcal{E}_{22}^{21}=-\widetilde{R}^{-1}-P_{z_{h}}^{-1} S_{z_{h}} P_{z_{h}}^{-1}$.

$$
\begin{align*}
& {\left[\begin{array}{cccc}
\mathcal{E}_{11}^{20} & S_{12} & P_{z}^{-1} \mathcal{B}+\tilde{R}^{-1}-S_{12} & P_{z}^{-1} B_{z z_{\tau}} G+H^{T} M \\
* & -\tilde{R}^{-1}-P_{z_{h}}^{-1} S_{z_{h}} P_{z_{h}}^{-1} & \tilde{R}^{-1}-S_{12}^{T} & 0 \\
* & * & \mathcal{E}_{33}^{20} & -H^{T} M \\
* & * & * & \nu(M)
\end{array}\right]} \\
& +h^{2}\left[\begin{array}{cccc}
A_{z z_{\tau}}^{T} \mathcal{R} A_{z z_{\tau}} & 0 & A_{z z_{\tau}}^{T} \mathcal{R B} & A_{z z_{\tau}}^{T} \mathcal{R} B_{z z_{\tau}} G \\
* & 0 & 0 & 0 \\
* & * & \mathcal{B}^{T} \mathcal{R B} & \mathcal{B}^{T} \mathcal{R} \mathcal{C} \\
* & * & * & \left(B_{z z_{\tau}} G\right)^{T} \mathcal{R} B_{z z_{\tau}} G
\end{array}\right] \leq 0 \tag{20}
\end{align*}
$$

$$
\left[\begin{array}{ccccc}
\mathcal{E}_{11}^{21} & S_{12} & P_{z}^{-1} \mathcal{B}+\tilde{R}^{-1}-S_{12} & P_{z}^{-1} B_{z z_{\tau}} G+H^{T} M & h A_{z z_{\tau}}^{T}  \tag{21}\\
* & \mathcal{E}_{22}^{21} & \tilde{R}^{-1}-S_{12}^{T} & 0 & 0 \\
* & * & \mathcal{E}_{33}^{21} & -H^{T} M & h \mathcal{B}^{T} \\
* & * & * & \nu(M) & h\left(B_{z z_{\tau}} G\right)^{T} \\
* & * & * & * & -P_{z} R_{z}^{-1} P_{z}
\end{array}\right] \leq 0
$$

Congruence with $\operatorname{diag}\left(\begin{array}{lllll}P_{z} & P_{z_{h}} & P_{z_{\tau}} & M^{-1} & I\end{array}\right)$ gives

$$
\left[\begin{array}{ccccc}
\mathcal{E}_{11}^{22} & P_{z} S_{12} P_{z_{h}} & \mathcal{E}_{13}^{22} & B_{z z_{\tau}} G M^{-1}+P_{z} H^{T} & h P_{z} A_{z z_{\tau}}^{T}  \tag{22}\\
* & \mathcal{E}_{22}^{22} & \mathcal{E}_{23}^{22} & 0 & 0 \\
* & * & \mathcal{E}_{33}^{22} & -P_{z_{\tau}} H^{T} & h P_{z_{\tau}} \mathcal{B}^{T} \\
* & * & * & M^{-1} \nu(M) M^{-1} & h M^{-1}\left(B_{z_{\tau}} G\right)^{T} \\
* & * & * & * & -P_{z} R_{z}{ }^{-1} P_{z}
\end{array}\right] \leq 0
$$

where $\mathcal{E}_{11}^{22}=P_{z} A_{z z_{\tau}}^{T}+A_{z z_{\tau}} P_{z}-\dot{P}_{z}+\epsilon P_{z}^{2}+S_{z}+Q_{z}-P_{z} \tilde{R}^{-1} P_{z}, \mathcal{E}_{13}^{22}=\mathcal{B} P_{z_{\tau}}+$ $P_{z} \tilde{R}^{-1} P_{z_{\tau}}-P_{z} S_{12} P_{z_{\tau}}, \mathcal{E}_{22}^{22}=-P_{z_{h}} \tilde{R}^{-1} P_{z_{h}}-S_{z_{h}}, \mathcal{E}_{23}^{22}=P_{z_{h}} \tilde{R}^{-1} P_{z_{\tau}}-P_{z_{h}} S_{12}^{T} P_{z_{\tau}}$, $\mathcal{E}_{33}^{22}=-(1-d) Q_{z_{\tau}}+P_{z_{\tau}}\left(-2 \tilde{R}^{-1}+S_{12}+S_{12}^{T}\right) P_{z_{\tau}}$.

Next, consider $\dot{P}_{z}$. One way to handle this term is similar to the approach in [43]. [43] considers bounded derivatives of the membership functions, i.e., $\left|\dot{q}_{i}\right| \leq \phi_{i}, i=1, \ldots, r$. In this paper, we only need lower bounds, $\phi_{i} \leq \dot{q}_{i}, i=1, \ldots, r$, where $\phi_{i}$ are known constants. Since $\sum_{i=1}^{k} \dot{q}_{i}(z)=0$, $-\sum_{i=1}^{r} \dot{q}_{i}(z)\left(P_{i}+X\right) \leq-\sum_{i=1}^{r} \phi_{i}\left(P_{i}+X\right)$ for any constant matrix $X$, which leads to $-\dot{P}_{z} \leq-P_{\phi}$, thus obtaining (13).

Remark 4. Note that we used the method above for simplicity, other possibilities to handle $\dot{P}_{z}$ are also available in the literature, see e.g. [44, 45].

Remark 5. If the condition in Theorem 1 holds with $Q=0$, then the system is stabilized for any variation of the delay.

Remark 6. Note that in (16) the reciprocal convexity was used, based on the approach in [42]. To further reduce the conservatism of (16), novel approaches have been developed in [46, 47]. However, in this paper, our main focus is on handling the nonlinear consequents, therefore we leave the aforementioned relaxation techniques to future work.

Due to the multiplication of several decision variables $P_{z} S_{12} P_{z_{h}}, P_{z} \tilde{R}^{-1} P_{z_{h}}$, etc, the obtained conditions are BMIs. In what follows we discuss different cases so that sufficient LMI conditions can be formulated.

### 3.2. LMI conditions

Next, we consider three cases for which sufficient LMI conditions can be developed:

### 3.2.1. Case 1: $S_{12}=0$

One reason due to which (13) is BMI is given by the terms containing $S_{12}$. If $S_{12}=0$ is imposed, we obtain:
Corollary 1. Consider the closed-loop system (11) and assume that $\dot{\tau} \leq d$, $d \in[0,1), \tau \leq h, h>0$, and there exists a domain $\mathcal{D}$ with $0 \in \mathcal{D}$, such that $\phi_{i} \leq \dot{q}_{i}(z), i=1, \ldots, r, \forall z \in \mathcal{D}$. The closed-loop system (11), with $d$, $h$ and $\phi_{i}, i=1, \ldots, r$, being known constants, is locally asymptotically stable in the largest Lyapunov level-set included in $\mathcal{D}$, if there exist matrices $P_{i}=P_{i}^{T}>0$, $X=X^{T}, P_{i}+X>0, R_{i}=R_{i}^{T}>0, \tilde{R}=\tilde{R}^{T}>0,\left[\begin{array}{cc}R_{i} & P_{i} \\ P_{i} & \tilde{R}\end{array}\right] \geq 0$, $S_{i}=S_{i}^{T}>0, Q_{i}=Q_{i}^{T}>0, \tilde{M}=\operatorname{diag}\left(m_{1}, \ldots, m_{p}\right)>0, N_{i}, i=1, \ldots, r$, and constant $\tilde{\epsilon}>0$ so that

$$
\begin{equation*}
F_{i j k l m}+F_{j i k l m}+F_{i j m l k}+F_{j i m l k}<0, \tag{23}
\end{equation*}
$$

with $F_{i j k l m}$ given as

$$
\begin{align*}
& F_{i j k l m}= \\
& {\left[\begin{array}{cccccccc}
\mathcal{E}_{11}^{24} & 0 & \mathcal{E}_{13}^{24} & B_{i k} G \tilde{M}+P_{i} H^{T} & h P_{i} A_{j k}^{T} & P_{i} & 0 & P_{i} \\
* & \mathcal{E}_{22}^{24} & 0 & 0 & 0 & 0 & P_{l} & 0 \\
* & * & \mathcal{E}_{33}^{24} & -P_{k} H^{T} & h\left(P_{m} D_{i k}^{T}-N_{m}^{T} B_{i k}^{T}\right) & P_{k} & P_{k} & 0 \\
* & * & * & \nu(\tilde{M}) & h \tilde{M}\left(B_{i k} G\right)^{T} & 0 & 0 & 0 \\
* & * & * & * & R_{i}-2 P_{i} & 0 & 0 & 0 \\
* & * & * & * & * & -\tilde{R} & 0 & 0 \\
* & * & * & * & * & * & -\tilde{R} & 0 \\
* & * & * & * & * & * & * & -\tilde{\epsilon} I
\end{array}\right]} \tag{24}
\end{align*}
$$

and

$$
\begin{aligned}
& \mathcal{E}_{11}^{24}=P_{i} A_{j k}^{T}+A_{j k} P_{i}-P_{\phi}+S_{i}+Q_{i}+2 \tilde{R}-4 P_{i}, \\
& \mathcal{E}_{13}^{24}=D_{i k} P_{m}-B_{i k} N_{m}, \quad \mathcal{E}_{22}^{24}=2 \tilde{R}-4 P_{l}-S_{l}, \\
& \mathcal{E}_{33}^{24}=-(1-d) Q_{k}+4 \tilde{R}-8 P_{k},
\end{aligned}
$$

The controller gains are computed as $K(z)=N_{z} P_{z}^{-1}$.

Proof. Consider (13) with $S_{12}=0$. In this case, (13) can be written as

$$
\begin{align*}
& {\left[\begin{array}{cccccc}
\mathcal{E}_{11}^{25} & 0 & D_{z z_{\tau}} P_{z_{\tau}}-B_{z z_{\tau}} N_{z_{\tau}} & B_{z z_{\tau}} G \tilde{M}+P_{z} H^{T} & h P_{z} A_{z z_{\tau}}^{T} \\
* & -S_{z_{h}} & 0 & 0 & 0 \\
* & * & -(1-d) Q_{z_{\tau}} & -P_{z_{\tau}} H^{T} & h P_{z_{\tau}} \mathcal{B}^{T} \\
* & * & * & \nu(\tilde{M}) & h \tilde{M}\left(B_{z z_{\tau}} G\right)^{T} \\
* & * & * & * & -P_{z} R_{z}{ }^{-1} P_{z}
\end{array}\right]} \\
& +\left[\begin{array}{ccccc}
-2 P_{z} \tilde{R}^{-1} P_{z} & 0 & 0 & 0 & 0 \\
* & -2 P_{z_{h}} \tilde{R}^{-1} P_{z_{h}} & 0 & 0 & 0 \\
* & * & -4 P_{z_{\tau}} \tilde{R}^{-1} P_{z_{\tau}} & 0 & 0 \\
* & * & * & 0 & 0 \\
* & * & * & * & 0
\end{array}\right] \\
& +\left[\begin{array}{ccccc}
P_{z} \tilde{R}^{-1} P_{z} & 0 & P_{z} \tilde{R}^{-1} P_{z_{\tau}} & 0 & 0 \\
* & P_{z_{h}} \tilde{R}^{-1} P_{z_{h}} & P_{z_{h}} \tilde{R}^{-1} P_{z_{\tau}} & 0 & 0 \\
* & * & 2 P_{z_{\tau}} \tilde{R}^{-1} P_{z_{\tau}} & 0 & 0 \\
* & * & * & 0 & 0 \\
* & * & * & * & 0
\end{array}\right] \leq 0 \tag{25}
\end{align*}
$$

with $\mathcal{E}_{11}^{25}=P_{z} A_{z z_{\tau}}^{T}+A_{z z_{\tau}} P_{z}-P_{\phi}+S_{z}+Q_{z}+\epsilon P_{z}^{2}$. Next, applying the Schur complement on (25) and on $\epsilon P_{z}^{2}$, denoting $\tilde{M}=M^{-1}$, letting $K(z(t-\tau))=$ $N_{z_{\tau}} P_{z_{\tau}}^{-1}, \tilde{\epsilon}=\frac{1}{\epsilon}$, and using Lemma 1 on the terms $-P_{z} \tilde{R}^{-1} P_{z},-2 P_{z} \tilde{R}^{-1} P_{z}$, $-4 P_{z} \tilde{R}^{-1} P_{z}$, we obtain

$$
\left[\begin{array}{cccccccc}
\mathcal{E}_{11}^{26} & 0 & \mathcal{E}_{13}^{26} & \mathcal{E}_{14}^{26} & h P_{z} A_{z z_{\tau}}^{T} & P_{z} & 0 & P_{z}  \tag{26}\\
* & \mathcal{E}_{22}^{26} & 0 & 0 & 0 & 0 & P_{z_{h}} & 0 \\
* & * & \mathcal{E}_{33}^{26} & -P_{z_{\tau}} H^{T} & h\left(P_{z_{\tau}} D_{z z_{\tau}}^{T}-N_{z_{\tau}}^{T} B_{z z_{\tau}}^{T}\right) & P_{z_{\tau}} & P_{z_{\tau}} & 0 \\
* & * & * & \nu(\tilde{M}) & h \tilde{M}\left(B_{z z_{\tau}} G\right)^{T} & 0 & 0 & 0 \\
* & * & * & * & R_{z}-2 P_{z} & 0 & 0 & 0 \\
* & * & * & * & * & -\tilde{R} & 0 & 0 \\
* & * & * & * & * & * & * & -\tilde{R} \\
* & * & * & * & * & * & -\tilde{\epsilon} I
\end{array}\right] \leq 0
$$

where

$$
\begin{aligned}
\mathcal{E}_{11}^{26} & =P_{z} A_{z z_{\tau}}^{T}+A_{z z_{\tau}} P_{z}-P_{\phi}+S_{z}+Q_{z}+2 \tilde{R}-4 P_{z}, \\
\mathcal{E}_{13}^{26} & =D_{z z_{\tau}} P_{z_{\tau}}-B_{z z_{\tau}} N_{z_{\tau}}, \\
\mathcal{E}_{14}^{26} & =B_{z z_{\tau}} G \tilde{M}+P_{z} H^{T} \mathcal{E}_{22}^{26}=2 \tilde{R}-4 P_{z_{h}}-S_{z_{h}}, \\
\mathcal{E}_{33}^{26} & =-(1-d) Q_{z_{\tau}}+4 \tilde{R}-8 P_{z_{\tau}}, \\
\nu(\tilde{M}) & =-2 \tilde{M} \operatorname{diag}\left(\frac{1}{b_{1}}, \ldots, \frac{1}{b_{r}}\right), \\
P_{\phi} & =\sum_{k=1}^{r} \phi_{k}\left(P_{k}+X\right) .
\end{aligned}
$$

Applying Lemma 2 on the double sums involving $z$ and $z_{\tau}$, we get (24).
Although the conditions developed are LMIs, they are quite conservative. Beside the relaxation used on (26), conservatism stems from

1. the choice of $S_{12}=0$, and
2. using Lemma 1 on the terms $-P_{z} \tilde{R}^{-1} P_{z},-2 P_{z} \tilde{R}^{-1} P_{z},-4 P_{z} \tilde{R}^{-1} P_{z}$

Thus, in what follows, we consider two other options.

### 3.2.2. Case 2: $R(s)=0$

The second case we consider is when the conditions do not depend on the maximum delay, i.e., without the term $h \int_{-h}^{0} \int_{t+\theta}^{t} \dot{x}^{T}(s) P(s)^{-1} R(s) P(s)^{-1} \dot{x}(s) d s d \theta$ in the Lyapunov functional (12).

For this case, we formulate the following corollary:
Corollary 2. Consider the closed-loop system (11) and assume that $\dot{\tau} \leq d$, $d \in[0,1)$, and there exists a domain $\mathcal{D}$ with $0 \in \mathcal{D}$, such that $\phi_{i} \leq \dot{q}_{i}(z)$, $i=1, \ldots, r, \forall z \in \mathcal{D}$. The closed-loop system (11), with $d$ and $\phi_{i}, i=1, \ldots, r$, being known constants, is locally asymptotically stable in the largest Lyapunov level-set included in $\mathcal{D}$, if there exist matrices $P_{i}=P_{i}^{T}>0, X=X^{T}$, $P_{i}+X>0$ and $Q_{i}=Q_{i}^{T}>0, S_{i}=S_{i}^{T}>0, \tilde{M}=\operatorname{diag}\left(m_{1}, \ldots, m_{p}\right)>0, N_{i}$, $i=1, \ldots, r$ and $\tilde{\epsilon}>0$ so that

$$
\begin{equation*}
F_{i j k l m}+F_{j i k l m}+F_{i j m l k}+F_{j i m l k}<0, \tag{27}
\end{equation*}
$$

where

$$
F_{i j k l m}=\left[\begin{array}{ccccc}
\mathcal{E}_{11}^{28} & 0 & D_{j k} P_{m}-B_{j k} N_{m} & B_{j k} G \tilde{M}+P_{i} H^{T} & P_{i}  \tag{28}\\
* & -S_{l} & 0 & 0 & 0 \\
* & * & -(1-d) Q_{k} & -P_{k} H^{T} & 0 \\
* & * & * & \nu(\tilde{M}) & 0 \\
* & * & * & * & -\tilde{\epsilon} I
\end{array}\right]
$$

and

$$
\begin{aligned}
\mathcal{E}_{11}^{28} & =P_{i} A_{j k}^{T}+A_{j k} P_{i}+S_{i}+Q_{i}-P_{\phi}, \\
\nu(\tilde{M}) & =-2 \tilde{M} \operatorname{diag}\left(\frac{1}{b_{1}}, \ldots, \frac{1}{b_{r}}\right),
\end{aligned}
$$

The gains are computed as $K(z)=N_{z} P_{z}^{-1}$.
Proof. Consider (12) with $R(s)=0$. Then, (15) becomes

$$
\begin{align*}
\dot{V}(t, x, \dot{x})= & x^{T}(t) P_{z}^{-1} \dot{x}(t)+\dot{x}^{T}(t) P_{z}^{-1} x(t)+x^{T}(t) \dot{P}_{z}^{-1} x(t) \\
& +x^{T}(t) P_{z}^{-1}\left[S_{z}+Q_{z}\right] P_{z}^{-1} x(t) \\
& -x^{T}(t-h) P_{z_{h}}^{-1} S_{z_{h}} P_{z_{h}}^{-1} x(t-h)  \tag{29}\\
& -(1-\dot{\tau}(t)) x^{T}(t-\tau) P_{z_{\tau}}^{-1} Q_{z_{\tau}} P_{z_{\tau}}^{-1} x(t-\tau)
\end{align*}
$$

and condition (21) in the proof of Theorem 1 is reduced to

$$
\left[\begin{array}{cccc}
\mathcal{E}_{11}^{30} & 0 & P_{z}^{-1} \mathcal{B} & P_{z}^{-1} B_{z z_{\tau}} G+H^{T} M  \tag{30}\\
* & -P_{z_{h}}^{-1} S_{z_{h}} P_{z_{h}}^{-1} & 0 & 0 \\
* & * & -(1-d) P_{z_{\tau}}^{-1} Q_{z_{\tau}} P_{z_{\tau}}^{-1} & -H^{T} M \\
* & * & * & \nu(M)
\end{array}\right] \leq 0,
$$

where $\mathcal{E}_{11}^{30}=A_{z z_{\tau}}^{T} P_{z}^{-1}+P_{z}^{-1} A_{z z_{\tau}}+P_{z}^{-1}\left(S_{z}+Q_{z}-P_{\phi}\right) P_{z}^{-1}+\epsilon I$.
Congruence of (30) with diag ( $\left.\begin{array}{lllll}P_{z} & P_{z_{h}} & P_{z_{\tau}} & M^{-1}\end{array}\right)$, gives

$$
\left[\begin{array}{cccc}
\mathcal{E}_{11}^{31} & 0 & \mathcal{B} P_{z_{\tau}} & B_{z_{\tau}} G M^{-1}+P_{z} H^{T}  \tag{31}\\
* & -S_{z_{h}} & 0 & 0 \\
* & * & -(1-d) Q_{z_{\tau}} & -P_{z_{\tau}} H^{T} \\
* & * & * & M^{-1} \nu(M) M^{-1}
\end{array}\right] \leq 0
$$

where $\mathcal{E}_{11}^{31}=P_{z} A_{z z_{\tau}}^{T}+A_{z z_{\tau}} P_{z}+S_{z}+Q_{z}-P_{\phi}+\epsilon P_{z}^{2}$. Next, using the Schur complement on $\epsilon P_{z}^{2}$, denoting $\tilde{M}=M^{-1}$, letting $K(z(t-\tau))=N_{z_{\tau}} P_{z_{\tau}}^{-1}$,
$\tilde{\epsilon}=\frac{1}{\epsilon}$, we obtain

$$
\left[\begin{array}{ccccc}
\mathcal{E}_{11}^{32} & 0 & D_{z z_{\tau}} P_{z_{\tau}}-B_{z z_{\tau}} N_{z_{\tau}} & B_{z z_{\tau}} G \tilde{M}+P_{z} H^{T} & P_{z}  \tag{32}\\
* & -S_{z_{h}} & 0 & 0 & 0 \\
* & * & -(1-d) Q_{z_{\tau}} & -P_{z_{\tau}} H^{T} & 0 \\
* & * & * & \nu(\tilde{M}) & 0 \\
* & * & * & * & -\tilde{\epsilon} I
\end{array}\right] \leq 0
$$

where $\mathcal{E}_{11}^{32}=P_{z} A_{z z_{\tau}}^{T}+A_{z z_{\tau}} P_{z}+S_{z}+Q_{z}-P_{\phi}$.
Relaxing the double sums containing $z$ and $z_{\tau}$ in (32) we get the condition (28).

For this case we do not use $R$, thus the result is independent of the maximum delay.

Remark 7. Although the term containing $S(s)$ can also be removed from the Lyapunov function, throughout the proof we kept it for consistency. In fact, condition (27) can be reduced to

$$
\begin{equation*}
F_{i j k m}+F_{j i k m}+F_{i j m k}+F_{j i m k}<0 \tag{33}
\end{equation*}
$$

where $F_{i j k m}$ is defined as:

$$
\begin{align*}
& F_{i j k m}= \\
& {\left[\begin{array}{cccc}
P_{i} A_{j k}^{T}+A_{j k} P_{i}+Q_{i}-P_{\phi} & D_{j k} P_{m}-B_{j k} N_{m} & B_{j k} G \tilde{M}+P_{i} H^{T} & P_{i} \\
* & -(1-d) Q_{k} & -P_{m} H^{T} & 0 \\
* & * & \nu(\tilde{M}) & 0 \\
* & * & * & -\tilde{\epsilon} I
\end{array}\right]} \tag{34}
\end{align*}
$$

Remark 8. Note that the results in [30] are a special case of Corollary 2, obtained by choosing $P_{i}=P$ and $Q_{i}=Q, i=1,2, \ldots, r$.

### 3.2.3. Case 3: Constant P

The final case we consider is when $P_{i}=P_{i}^{T}=P, i=1, \ldots, r$, i.e., the first term in $V(t)$ is quadratic in $x$ and $\dot{P}_{z}=\dot{P}=0$. Then, we have:

Corollary 3. The closed-loop system (11), with $\dot{\tau} \leq d, d \in[0,1), \tau \leq$ $h, h>0, d$ and $h$ known constants is asymptotically stable, if there exist $P=P^{T}>0, R_{i}=R_{i}^{T}>0, \tilde{\tilde{R}}=\tilde{\tilde{R}}^{T}>0, R_{i} \geq \tilde{\tilde{R}}, \tilde{S}_{12},\left[\begin{array}{cc}\tilde{\tilde{R}} & \tilde{S}_{12} \\ \tilde{S}_{12}^{T} & \tilde{\tilde{R}}\end{array}\right] \geq 0$, $S_{i}=S_{i}^{T}>0, Q_{i}=Q_{i}^{T}>0, \tilde{M}=\operatorname{diag}\left(m_{1}, \ldots, m_{p}\right)>0, N_{i}, i=1, \ldots, r$ and $\tilde{\epsilon}>0$ such that

$$
\begin{equation*}
F_{i j k l}+F_{i k j l}<0, \tag{35}
\end{equation*}
$$

where

$$
\begin{align*}
& F_{i j k l}= \\
& {\left[\begin{array}{cccccc}
\mathcal{E}_{11}^{36} & \tilde{S}_{12} & \mathcal{E}_{13}^{36} & B_{i j} G \tilde{M}+P H^{T} & h P A_{i j}^{T} & P \\
* & -\tilde{R}-S_{l} & \tilde{\tilde{R}}^{T}-\tilde{S}_{12}^{T} & 0 & 0 & 0 \\
* & * & \mathcal{E}_{11}^{36} & -P H^{T} & h\left(P D_{i j}^{T}-N_{k}^{T} B_{i j}^{T}\right) & 0 \\
* & * & * & \nu(\tilde{M}) & h \tilde{M}\left(B_{i j} G\right)^{T} & 0 \\
* & * & * & * & R_{i}-2 P & 0 \\
* & * & * & * & * & -\tilde{\epsilon} I
\end{array}\right]} \tag{36}
\end{align*}
$$

and

$$
\begin{aligned}
\mathcal{E}_{11}^{36} & =P A_{i j}^{T}+A_{i j} P+S_{i}+Q_{i}-\tilde{\tilde{R}}, \\
\mathcal{E}_{13}^{36} & =D_{i j} P-B_{i j} N_{k}+\tilde{\tilde{R}}-\tilde{S}_{12}, \\
\mathcal{E}_{11}^{36} & =-(1-d) Q_{k}-2 \tilde{\tilde{R}}+\tilde{S}_{12}+\tilde{S}_{12}^{T}, \\
\nu(\tilde{M}) & =-2 \tilde{M} \operatorname{diag}\left(\frac{1}{b_{1}}, \ldots, \frac{1}{b_{r}}\right),
\end{aligned}
$$

The controller gains are computed as $K(z)=N_{z} P^{-1}$.

Proof. Consider (13) with $P_{i}=P$. Using the Schur complement on $\epsilon P^{2}$, denoting $P \tilde{R}^{-1} P=\tilde{\tilde{R}}, \tilde{M}=M^{-1}, P S_{12} P=\tilde{S}_{12}$, letting $K(z(t-\tau))=$ $N_{z_{\tau}} P_{z_{\tau}}^{-1}, \tilde{\epsilon}=\frac{1}{\epsilon}$, and using $-P R_{z}^{-1} P \leq R_{z}-2 P$, we obtain (37), where $\mathcal{E}_{11}^{37}=P A_{z z_{\tau}}^{T}+A_{z z_{\tau}} P+S_{z}+Q_{z}-\tilde{\tilde{R}}, \mathcal{E}_{13}^{37}=D_{z z_{\tau}} P-B_{z z_{\tau}} N_{z_{\tau}}+\tilde{\tilde{R}}-\tilde{S}_{12}$, $\mathcal{E}_{33}^{37}=-(1-d) Q_{z_{\tau}}-2 \tilde{\tilde{R}}+\tilde{S}_{12}+\tilde{S}_{12}^{T}$. Since $P \tilde{R}^{-1} P=\tilde{\tilde{R}}$, we have $R(s) \geq \tilde{\tilde{R}}$,
which is satisfied if $R_{i} \geq \tilde{\tilde{R}}$. Relaxing the double sum in

$$
\left[\begin{array}{cccccc}
\mathcal{E}_{11}^{37} & \tilde{S}_{12} & \mathcal{E}_{13}^{37} & B_{z z_{\tau}} G \tilde{M}+P H^{T} & h P A_{z z_{\tau}}^{T} & P  \tag{37}\\
* & -\tilde{\tilde{R}}-S_{z_{h}} & \tilde{\tilde{R}}-\tilde{S}_{12}^{T} & 0 & 0 & 0 \\
* & * & \mathcal{E}_{33}^{37} & -P H^{T} & h P \mathcal{B}^{T} & 0 \\
* & * & * & \nu(\tilde{M}) & h \tilde{M}\left(B_{z z_{\tau}} G\right)^{T} & 0 \\
* & * & * & * & R_{z}-2 P & 0 \\
* & * & * & * & * & -\tilde{\epsilon} I
\end{array}\right] \leq 0,
$$

we get condition (35).

Remark 9. Since $P$ is constant, $\dot{P}_{z}=\sum_{i=1}^{r} \dot{q}_{i}(z) P=0$, and there is no need for $\phi_{i}$. Although the conditions with $P=P_{i}$ are theoretically more conservative than those of Corollary 1, due to the way LMI conditions are obtained, the two sets of conditions will be complementary.

Remark 10. The results of [31] are a special case of Corollary 3, obtained by choosing $Q_{i}=Q, R_{i}=R$, and $S_{i}=S, i=1,2, \ldots, r$.

It should be emphasized that the developed LMI conditions are by definition conservative. First of all, more complex Lyapunov-Krasovskii functionals could be used, and second, the conditions are only sufficient. Next to these, sources of conservativeness are: in Case 1, the handling of $\dot{P}_{z}$, in Case 2 the maximum delay is no longer taken into account, and in Case 3, the constant $P$ matrix. Furthermore, more advanced relaxations, e.g. [39, 40, 41] could be used to obtain LMI conditions.

### 3.3. Computational complexity

To solve the LMI conditions using the interior-point method, a good approximation [37] of the computational complexity is $\mathcal{O}\left(N_{d}^{2.1} N_{l}^{1.2}\right)$, with $N_{d}$ being the number of scalar decision variables and $N_{l}$ the row size of the LMI problem. In case of Corollaries 1,2 and 3 , these $N_{d}$ and $N_{l}$ are given by

- Corollary 1 :

$$
\begin{aligned}
N_{d} & =n_{x}\left(n_{x}+1\right)(2 r+1)+n_{x} n_{u} r+p+1 \\
N_{l} & =2 n_{x} r^{3}(r+1)^{2}
\end{aligned}
$$

- Corollary 2 :

$$
\begin{aligned}
N_{d} & =\frac{3 r n_{x}\left(n_{x}+1\right)}{2}+n_{x} n_{u} r+p+1 \\
N_{l} & =n_{x} r^{2}(r+1)^{2}
\end{aligned}
$$

- Corollary 3 :

$$
\begin{aligned}
& N_{d}=\frac{n_{x}(3 r+2)\left(n_{x}+1\right)}{2}+n_{x}^{2}+n_{x} n_{u} r+p+1 \\
& N_{l}=6 n_{x} r^{3}
\end{aligned}
$$

## 4. Examples and discussion

In the following, we illustrate and discuss the developed conditions.

### 4.1. Numerical example

To illustrate the difference between the design conditions we consider the system:

$$
\begin{align*}
& {\left[\begin{array}{l}
\dot{x}_{1} \\
\dot{x}_{2}
\end{array}\right]=\left[\begin{array}{cc}
-4 & -0.5 \\
0 & -4-\cos \left(x_{1}\right)
\end{array}\right]\left[\begin{array}{l}
x_{1} \\
x_{2}
\end{array}\right]} \\
& +\left[\begin{array}{cc}
1 & 2 \\
1.75+0.25 \cos \left(x_{1}\right) & 4+\cos \left(x_{1}(t-\tau)\right)
\end{array}\right]\left[\begin{array}{l}
x_{1}(t-\tau) \\
x_{2}(t-\tau)
\end{array}\right] \\
& +\left[\begin{array}{c}
0 \\
0.75+0.25 \cos \left(x_{1}\right)
\end{array}\right] u(t-\tau)  \tag{38}\\
& +\left[\begin{array}{c}
0 \\
-0.375-0.125 \cos \left(x_{1}\right)
\end{array}\right]\left(\alpha_{1}\left(x_{1}\right)+\alpha_{2}\left(x_{2}\right)\right)
\end{align*}
$$

where the nonlinear functions $\alpha_{1}\left(x_{1}\right)$ and $\alpha_{2}\left(x_{2}\right)$ satisfy Assumption 1 with $b_{1}=b_{2}=2$. For the simulations we will use $\alpha_{1}(v)=\alpha_{2}(v)=\cos ^{2}(v)+v$.

The equivalent TS model is given by:

$$
\begin{align*}
A_{11} & =A_{12}=\left[\begin{array}{cc}
-4 & -0.5 \\
0 & -3
\end{array}\right], \quad A_{21}=A_{22}=\left[\begin{array}{cc}
-4 & -0.5 \\
0 & -5
\end{array}\right] \\
B_{11} & =B_{12}=\left[\begin{array}{c}
0 \\
0.5
\end{array}\right], \quad B_{21}=B_{22}=\left[\begin{array}{l}
0 \\
1
\end{array}\right], \\
D_{11} & =\left[\begin{array}{cc}
1 & 2 \\
1.5 & 3
\end{array}\right], D_{12}=\left[\begin{array}{cc}
1 & 2 \\
1.5 & 5
\end{array}\right], G=\left[\begin{array}{ll}
-0.5 & -0.5
\end{array}\right]  \tag{39}\\
D_{21} & =\left[\begin{array}{ll}
1 & 2 \\
2 & 3
\end{array}\right], \quad D_{22}=\left[\begin{array}{cc}
1 & 2 \\
2 & 5
\end{array}\right], \quad H=\left[\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right] \\
q_{1}(z) & =\frac{1-\cos (z)}{2}, \quad q_{2}(z)=1-q_{1}(z), \quad z=x_{1} .
\end{align*}
$$

Note that the system matrices $A_{11}, A_{12}, A_{21}$, and $A_{22}$ are Hurwitz, but due to the delay terms the overall open-loop system is unstable.

In the following, we compare the three sets of LMI conditions developed in Section 3, i.e., the cases when $S_{12}=0, R_{i}=0$, and $P_{i}=P$ w.r.t. the maximum allowable $d$ and $h$. For this particular case $\dot{q}_{1}=\frac{1}{2} \sin \left(x_{1}\right) \dot{x}_{1}$. The conditions of Corollaries 1 and 2 have been applied with $\phi_{1}=\phi_{2}=-2.5$ and the results are only valid for the set of the initial states that is in the largest Lyapunov level-set included in $\mathcal{D}=\left\{\left(x_{1}, x_{2}\right) \left\lvert\,-2.5 \leq \frac{1}{2} \sin \left(x_{1}\right) \dot{x}_{1}\right.\right\}$.

The feasible results obtained using Corollary 1, Corollary 2 and Corollary 3 are illustrated in Fig. 2(a), as follows:

- Corollary 1: as discussed in Section 3, the LMI conditions are very conservative.
- Corollary 2: this approach gives feasible solutions for slowly-varying delays, $\dot{\tau} \leq 0.73$, for any $h$.
- Corollary 3: the results depend both on the delay and its derivative. Note that in this case $\phi$ is not needed and the results hold globally. Feasible solutions have been obtained for a larger domain than with Corollary 1.

For this particular example, the number of decision variables and row dimensions of the LMIs are given in Table 1.

Table 1: Computational complexity - numerical example

|  | Corollary 1 | Corollary 2 | Corollary 3 |
| :---: | :---: | :---: | :---: |
| $N_{d}$ | 37 | 25 | 35 |
| $N_{l}$ | 288 | 72 | 96 |

Next we simulate the system for: $\tau(t)=0.35+0.25 \cos (2 t)$. For this particular case, Corollary 3 provided the solution

$$
\begin{aligned}
N_{1} & =\left[\begin{array}{ll}
1.18 & 1.39
\end{array}\right], \quad N_{2}=\left[\begin{array}{ll}
1 & 2.33
\end{array}\right], \\
P & =\left[\begin{array}{cc}
0.52 & -0.07 \\
-0.07 & 0.35
\end{array}\right]
\end{aligned}
$$

The initial point is $x_{0}=\left[\begin{array}{ll}1 & 2\end{array}\right]^{T}$. The computed controller stabilizes system (38), see Fig. 2(b).


Figure 2: Results for the numerical example
A similar result has been obtained using Corollary 1. The trajectory of the closed-loop states is shown in Figure 3(a). As it can be seen in Figure $3(\mathrm{~b})$, the condition on the derivatives of the membership functions is satisfied for this particular trajectory.

Remark 11. Sufficient LMI conditions for multiple sum negativity problem is obtained, for simplicity, in this paper by the relaxation (9). Note however,


Figure 3: Results using Corollary 1
that LMI conditions can be obtained using more advanced relaxation technique, see e.g., [39, 40, 41]. Conditions obtained using such relaxations may significantly outperform the LMI conditions as stated in this paper.

Next, we compare the results presented in this paper with our previous works. As already stated in Remarks 8 and 10, the results in [30] are a special case of Corollary 2, while the conditions in [31] are a special case of Corollary 3. Thus, the solutions obtained by Corollaries 2 and 3 are more general than the previous results. This is confirmed by the simulations, as illustrated in Figure 4. In both cases, the maps obtained by the proposed conditions are superior to those in the previous publications.

In what follows, we compare our approach to that of [29]. If $\psi(v)$ is also sector-bounded with $b=2$, we can apply the conditions of Corollary 1 in [29]. The conditions of [29] are delay-dependent and can only handle fixed delays, while we assume that delays are time-varying. Therefore, for the comparison, we consider fixed delays, i.e. $d=0$.

We test the approaches on the time-delayed T-S model (6), with $h=$ $0.535, d=0$, the local matrices (39) and

$$
\begin{aligned}
D_{11} & =\left[\begin{array}{ll}
a_{1} & 2 \\
1.5 & 3
\end{array}\right], D_{12}=\left[\begin{array}{ll}
a_{1} & 2 \\
1.5 & 5
\end{array}\right], G=\left[\begin{array}{ll}
a_{2} & -0.5
\end{array}\right], \\
D_{21} & =\left[\begin{array}{ll}
a_{1} & 2 \\
2 & 3
\end{array}\right], D_{22}=\left[\begin{array}{cc}
a_{1} & 2 \\
2 & 5
\end{array}\right],
\end{aligned}
$$

where $a_{1}, a_{2} \in[-6,6]$ are two parameters. We analyze the points $\left(a_{1}, a_{2}\right)$ for


(a) Feasible solutions, '.' - Theorem 1 (b) Feasible solutions, 'o' - Theorem 1 from [30], $\qquad$ - Corollary 2

Figure 4: Comparison with previous work


Figure 5: Results for the numerical example: Feasible solutions, 'o'-Corollary 3, 'x'Corollary 1 from [29]
which feasible solutions can be obtained using Corollary 3 and Corollary 1 of [29], respectively. We assume that $\psi(v)$ satisfies Assumption 1 and at the same time it is sector-bounded, both bounds being $b=2$. Since we are only looking for the domain of feasible solutions, the exact form of the function $\psi(\cdot)$ is not important.

Fig. 5 shows a map of feasible solutions. The two results are complementary and they do not include in each other, i.e., Corollary 1 in [29] gives feasible solutions for $a_{1} \in[-2.8,4], a_{2} \in[-6,6]$, while the conditions of Corollary 3 are feasible for a different set of points.

### 4.2. BLDCM system

Next, we illustrate the application of Corollary 3 on a realistic application. We consider the following model adapted from [48] of a brushless DC motor (BLDCM) with time-delay:

$$
\begin{align*}
\dot{x}_{1} & =\frac{1}{\tau_{1}}\left(\sigma x_{2}+\rho x_{2} x_{3}-\eta x_{1}+\sin \left(x_{1}(t-\tau)\right)\right) \\
\dot{x}_{2} & =\frac{1}{\tau_{2}}\left(-x_{2}-x_{1}-x_{1} x_{3}+u_{q}+x_{1}(t-\tau) x_{2}(t-\tau)\right)  \tag{40}\\
\dot{x}_{3} & =\frac{1}{\tau_{3}}\left(x_{1} x_{2}-x_{3}+u_{d}+\sin \left(x_{3}\right)\right)
\end{align*}
$$

where $x_{1}, x_{2}$, and $x_{3}$ are the system states, $u_{q}$ and $u_{d}$ are the control inputs. The model parameters are adopted from [48]: $\tau_{1}=1, \tau_{2}=6.45, \tau_{3}=7.125$, $\sigma=16, \rho=1.516$, and $\eta=3$.

If one constructs a classic T-S model for (40), $x_{3}$ has to be used as a scheduling variable. In our approach the term $\sin \left(x_{3}\right)$ will be part of the nonlinear consequent, thereby reducing the number of local models. Since $\sin \left(x_{3}\right)$ does not fulfill Assumption 1, we use $\psi(H x)=\sin \left(x_{3}\right)+x_{3} . \psi(H x)$ satisfies Assumption 1 with $b=2$ and (40) can be rewritten as:

$$
\begin{align*}
{\left[\begin{array}{l}
\dot{x}_{1} \\
\dot{x}_{2} \\
\dot{x}_{3}
\end{array}\right] } & =\left[\begin{array}{ccc}
-\frac{\eta}{\tau_{1}} & \frac{\sigma+\rho x_{2}}{\tau_{1}} & 0 \\
-\frac{1}{\tau_{2}} & -\frac{1}{\tau_{2}} & -\frac{x_{1}}{\tau_{2}} \\
\frac{x_{2}}{\tau_{3}} & 0 & -\frac{2}{\tau_{3}}
\end{array}\right]\left[\begin{array}{l}
x_{1} \\
x_{2} \\
x_{3}
\end{array}\right] \\
& +\left[\begin{array}{ccc}
-\frac{\sin \left(x_{1}(t-\tau)\right)}{\tau_{1} x_{1}(t-\tau)} & 0 & 0 \\
0 & -\frac{x_{1}(t-\tau)}{\tau_{2}} & 0 \\
0 & 0 & 0
\end{array}\right]\left[\begin{array}{l}
x_{1}(t-\tau) \\
x_{2}(t-\tau) \\
x_{3}(t-\tau)
\end{array}\right]  \tag{41}\\
& +\left[\begin{array}{ll}
0 & 0 \\
1 & 0 \\
0 & 1
\end{array}\right]\left[\begin{array}{l}
u_{q}(t-\tau) \\
u_{d}(t-\tau)
\end{array}\right]+\left[\begin{array}{ll}
0 & 0 \\
1 & 0 \\
0 & 1
\end{array}\right]\left[\begin{array}{l}
0 \\
\frac{1}{\tau_{3}}
\end{array}\right]\left(\sin \left(x_{3}\right)+x_{3}\right)
\end{align*}
$$

Equation (41) has a form similar to (6). In order to obtain the T-S model we use the sector nonlinearity approach, which leads to 8 rules for the current and 8 for the past scheduling variables. To determine the local models we
assume $x_{1}, x_{2} \in[-1,1]$. Using the sector nonlinearity approach, we have:

$$
\begin{align*}
& \underline{w}_{1}=\frac{1-x_{1}}{2}, \quad \bar{w}_{1}=1-\underline{w}_{1}, \quad \underline{w}_{2}=\frac{1-x_{2}}{2}, \quad \bar{w}_{2}=1-\underline{w}_{2} \\
& \underline{w}_{3}=\frac{x_{1}-\sin \left(x_{1}\right)}{0.16 x_{1}}, \quad \bar{w}_{3}=1-\underline{w}_{3} \tag{42}
\end{align*}
$$

and we construct $r=2^{3}$ membership functions as:

$$
\begin{array}{llll}
q_{1}=\underline{w}_{1} \underline{w}_{2} \underline{w}_{3}, & q_{2}=\underline{w}_{1} \underline{w}_{2} \bar{w}_{3}, & q_{3}=\underline{w}_{1} \bar{w}_{2} \underline{w}_{3}, & q_{4}=\underline{w}_{1} \bar{w}_{2} \bar{w}_{3}, \\
q_{5}=\bar{w}_{1} \underline{w}_{2} \underline{w}_{3}, & q_{6}=\bar{w}_{1} \underline{w}_{2} \bar{w}_{3}, & q_{7}=\bar{w}_{1} \bar{w}_{2} \underline{w}_{3}, & q_{8}=\bar{w}_{1} \bar{w}_{2} \bar{w}_{3} . \tag{43}
\end{array}
$$

Note that these membership functions are evaluated both for the current and delayed time instant. Some of the local models are:

$$
\begin{align*}
A_{1 j} & =\left[\begin{array}{ccc}
-3 & 14.48 & 0 \\
-0.15 & -0.15 & 0.15 \\
-0.14 & 0 & -0.28
\end{array}\right], \quad A_{8 j}=\left[\begin{array}{ccc}
-3 & 17.51 & 0 \\
-0.15 & -0.15 & -0.15 \\
0.14 & 0 & -0.28
\end{array}\right] \\
D_{i 1} & =\left[\begin{array}{ccc}
0.84 & 0 & 0 \\
0 & -0.15 & 0 \\
0 & 0 & 0
\end{array}\right], \quad D_{i 8}=\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & 0.15 & 0 \\
0 & 0 & 0
\end{array}\right],  \tag{44}\\
B & =\left[\begin{array}{ll}
0 & 0 \\
1 & 0 \\
0 & 1
\end{array}\right], \quad G=\left[\begin{array}{c}
0 \\
0.14
\end{array}\right], \quad H=\left[\begin{array}{lll}
0 & 0 & 1
\end{array}\right], \\
& i, j=1,2, \cdots, 8
\end{align*}
$$

We test Corollary 3 on our model. The domain for the maximum delay and maximum variation of delay for which this approach gives feasible solutions is presented in Fig. 6(a).

Next we assume that the delay is given by $\tau(t)=0.25+0.25 \cos (2 t)$ and use the controller gains

$$
\begin{array}{lll}
K_{1}=\left[\begin{array}{ccc}
0.01 & 0.63 & 0.03 \\
-0.04 & -0.2 & 0.66
\end{array}\right], & K_{2}=\left[\begin{array}{ccc}
0.01 & 0.64 & 0.03 \\
-0.04 & -0.2 & 0.66
\end{array}\right], \\
K_{3}=\left[\begin{array}{ccc}
0.01 & 0.67 & 0.03 \\
-0.01 & -0.1 & 0.66
\end{array}\right], & K_{4}=\left[\begin{array}{ccc}
0.02 & 0.69 & 0.03 \\
0 & -0.07 & 0.66
\end{array}\right], \\
K_{5}=\left[\begin{array}{ccc}
0.02 & 0.97 & 0.01 \\
-0.02 & -0.02 & 0.74
\end{array}\right], & K_{6}=\left[\begin{array}{ccc}
0.02 & 0.97 & 0 \\
-0.03 & 0 & 0.75
\end{array}\right],  \tag{45}\\
K_{7}=\left[\begin{array}{ccc}
0.02 & 0.97 & 0 \\
-0.03 & 0.04 & 0.77
\end{array}\right], & K_{8}=\left[\begin{array}{ccc}
0.03 & 1 & 0 \\
-0.02 & 0.13 & 0.78
\end{array}\right],
\end{array}
$$



Figure 6: Results for the BLDCM system

The trajectories of the states of the closed-loop system, starting from the initial condition $x_{0}=\left[\begin{array}{lll}0.5 & 0.25 & 0.25\end{array}\right]^{T}$ are given in Fig. 6(b). As it can be seen, the computed controller stabilizes the system.

## 5. Conclusions

This paper presented stabilization conditions for time-delayed nonlinear systems, under the assumption that the delay affects both the states and the input. The system was represented by a T-S model with slope-bounded nonlinear consequents. Sufficient conditions were formulated as linear matrix inequalities for three cases. These have been illustrated on a numerical example and compared to another approach from the literature. In the future we will consider observer design, observer-based control, $H_{\infty}$ control, and also removing the matching condition. We will also consider several different delays and more complex Lyapunov functionals.

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